A PROOF OF THE TWIN PRIME CONJECTURE AND OTHER POSSIBLE APPLICATIONS

by

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ABSTRACT :

An elementary proof of the Twin Prime Conjecture (TPC) is given.
A definite integral expression is obtained for $\pi_2(N)$; applying the Riemann definition of the integral, it is eventually shown that this integral is unbounded, proving TPC.

We also indicate that there exist infinitely many prime pairs (p, q) with $p < q$ such that : $q - p = 2k$, $k = 1, 2, \ldots$ and $q - 2p = 1$; also, we indicate that there exist infinitely many prime triplets (p, p+2u, p+2u+2v), provided u and v are chosen so as to avoid trivial cases.

KEY WORDS OR TERMS

asymptotic behavior ; counting function ; prime ; Twin Prime Conjecture ; twin prime pairs ; m-tuplets of primes with specified differences ; Sophie Germain prime pairs.

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1. INTRODUCTION :

The primary purpose of this paper is to prove the Twin Prime Conjecture (TPC), which states that the number of "twin prime" pairs (prime pairs (p, q) such that q = p+2) is infinite. In subsequent sections, we also prove a number of similar and more general conjectures.

We let $\delta(n)$ denote the characteristic function of the odd primes, where n is any positive integer. This notation excludes the prime 2, which makes the ensuing analysis a bit more convenient. We also introduce the polynomial functions f_N and $g_{2:N}$, defined as follows for N = 3, 4, … :

$$f_N(z) = \sum_{n=3}^{N} \delta(n)\, z^n \tag{1}$$

$$g_{2:N}(z) = \sum_{n=3}^{N} \delta(n)\delta(n+2)\, z^n \quad . \tag{2}$$

Clearly, f_N and $g_{2:N}$ are polynomials, hence entire functions.

By the Cauchy integral formula, it follows that for all $n \le N$:

$$\delta(n) = \frac{1}{2i\pi} \oint_\Gamma \frac{f_N(z)}{z^{n+1}}\, dz \quad . \tag{3}$$

Here, Γ is any simple closed contour in the complex plane, taken in the positive (i.e. counter-clockwise) direction and containing the origin within its interior.

We also let x be any real value in the unit interval I ≡ [0,1].

We may then express $g_2(x)$ in terms of f as follows :

$$g_{2:N}(x) = \sum_{n=3}^{N} \delta(n)\, x^n \,\frac{1}{2i\pi}\oint_\Gamma \frac{f_N(z)}{z^{n+3}}\,dz = \frac{1}{2i\pi}\oint_\Gamma \frac{f_N(z)}{z^3}\,dz \sum_{n=3}^{N}\delta(n)\left(\frac{x}{z}\right)^n \text{, or :}$$

$$g_{2:N}(x) = \frac{1}{2i\pi}\oint_\Gamma \frac{f_N(z)\,f_N(x/z)}{z^3}\,dz \quad . \qquad (4)$$

We may suppose that Γ is the circle {z = r exp(iθ), 0 ≤ θ < 2π}, where 0 < r ≤ 1 . Also,

assume that $x = r^2$, so that $\left|\frac{x}{z}\right| = r \in I$. Then $g_{2:N}(r^2) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_N(re^{i\theta})\,f_N(re^{-i\theta})}{r^2 e^{2i\theta}}\,d\theta$.
Since $g_{2:N}(r^2)$ must be real-valued, we then see that

$$r^2\, g_{2:N}(r^2) = \frac{1}{2\pi}\int_0^{2\pi} \cos 2\theta\, \left|f_N(re^{i\theta})\right|^2 d\theta \quad . \qquad (5)$$

Now $\left|f_N(re^{i\theta})\right|^2 = f_N(re^{i\theta})\, f_N(re^{-i\theta}) = \sum_{m=3}^{N}\sum_{n=3}^{N} r^{m+n}\,\delta(m)\,\delta(n)\,e^{i\theta(m-n)}$, hence :

$$\left|f_N(re^{i\theta})\right|^2 = \sum_{m=3}^{N}\sum_{n=3}^{N} r^{m+n}\,\delta(m)\,\delta(n)\cos\{(m-n)\theta\} \quad . \qquad (6)$$

Note that $\left|f_N(re^{i\theta})\right|^2 \le \left|f_N(r)\right|^2 \le \left|f_N(1)\right|^2 = \{\pi^*(N)\}^2$, where $\pi^*(N) = \pi(N) - 1$.

Since $f_N(re^{i\theta}) = \sum_{m=3}^{N} \delta(m)\, r^m \cos m\theta + i \sum_{m=3}^{N} \delta(m)\, r^m \sin m\theta$, an alternative mode of expression that avoids double sums is the following :

$$\left|f_N(re^{i\theta})\right|^2 = \left\{\sum_{m=3}^{N}\delta(m)\, r^m \cos m\theta\right\}^2 + \left\{\sum_{m=3}^{N}\delta(m)\, r^m \sin m\theta\right\}^2 \qquad (7)$$

We make the definition :

$$F_N(\theta) = \left|f_N(e^{i\theta})\right|^2 = \left\{\sum_{3\le p\le N}\cos p\theta\right\}^2 + \left\{\sum_{3\le p\le N}\sin p\theta\right\}^2 \qquad (8)$$

Also, using (6) :

$$F_N(\theta) = \sum_{3\le p\le N}\sum_{3\le q\le N}\cos\{(p-q)\theta\} \qquad (9)$$

It is found by integrating the expression in (9) that $\int_0^{\pi} F_N(\theta)\,d\theta = \pi(\pi^*(N))$.

Since $F_N(\pi - \theta) = F_N(\theta)$, we see that

$$\frac{2}{\pi}\int_0^{\pi/2} F_N(\theta)\, d\theta = \pi^*(N) \qquad (10)$$

We now make the following definition :

$$F_N\theta) + F_N(\pi/2 - \theta) = J_N(\theta) \qquad (11)$$

It then follows that :

$$\frac{2}{\pi}\int_0^{\pi/4} J_N(\theta)\, d\theta = \pi^*(N) \qquad (12)$$

2. <u>MORE DEFINITE INTEGRAL EXPRESSIONS</u> :

Consider the following special values :

$$\pi^*(N) = f_N(1) = \sum_{n=3}^{N} \delta(n) \qquad ; \qquad (13)$$

$$\pi_2(N) = g_{2:N}(1) =. \sum_{n=3}^{N} \delta(n)\,\delta(n+2) \qquad (14)$$

We may set r = 1 in (5), and obtain :

$$\pi_2(N) = \frac{1}{2\pi}\int_0^{2\pi} \cos 2\theta\; F_N(\theta)\; d\theta\ . \qquad (15)$$

Our goal is to show that the integral in (15) is an unbounded function of N . Just as was done for the integral in (10), we may decompose the integral in (15) into a pair of integrals, using the relations : $\cos 2(2\pi - \theta) = \cos 2\theta$, $F_N(2\pi - \theta) = F_N(\theta)$. Then $\pi_2(N) = \frac{1}{\pi}\int_0^{\pi} \cos 2\theta\; F_N(\theta)\; d\theta$.

We repeat the process, using the relations : $\cos 2(\pi - \theta) = \cos 2\theta$, $F_N(\pi - \theta) = F_N(\theta)$; this yields :

$$\pi_2(N) = \frac{2}{\pi}\int_0^{\pi/2} \cos 2\theta\; F_N(\theta)\; d\theta \qquad (16)$$

Now make the following definition :

$$F_N\theta) - F_N(\pi/2 - \theta) = G_N(\theta) \qquad (17)$$

Using the relation $\cos 2(\pi/2 - \theta) = -\cos 2\theta$, and the definition in (17), we obtain :

$$\pi_2(N) = \frac{2}{\pi}\int_0^{\pi/4} \cos 2\theta \, G_N(\theta)\, d\theta \qquad (18)$$

It is in this latter form that we will obtain the order of magnitude of $\pi_2(N)$.

Clearly, the values of cos 2θ over the interval [0, π/4] are non-negative and decreasing.

It would therefore be helpful if we could determine the behavior of the function $G_N(\theta)$ over this same interval.

3. <u>ESTIMATES OF INTEGRALS</u> :

The following estimate for the integral in (10) is appropriate for the Riemann definition of the integral :

$$\pi^*(N;n) = \frac{1}{n}\sum_{k=0}^{n} F_N(k\pi/2n) \qquad (19)$$

Using the same reasoning on the integral in (12), we may also define

$$\pi^{\#}(N\,;\,n) = \frac{1}{2n}\sum_{k=0}^{n} J_N(k\pi/4n) \qquad (20)$$

We find that if n is replaced by 2n in (19), we obtain the following relation, after simplification :

$$\pi^*(N;\,2n) = \pi^{\#}(N\,;\,n) - \frac{F_N(\pi/4)}{2n} \qquad (21)$$

Thus, the definitions in (19) and (20) are essentially equivalent for large n .

On the other hand, based on the same reasoning applied to the integrals in (16) and (18), make the following definitions :

$$\pi_2^*(N;n) = \frac{1}{n}\sum_{k=0}^{n} \cos(k\pi/n)\, F_N(k\pi/2n)\ ; \qquad (22)$$

$$\pi_2^{\#}(N;n) = \frac{1}{2n}\sum_{k=0}^{n} \cos(k\pi/2n)\, G_N(k\pi/4n)\ ; \qquad (23)$$

Note the following :

$$\pi^*(N\,;\,n) \sim \pi^{\#}(N\,;\,n) \sim \pi^*(N)\ ;\ \pi_2^*(N;n) \sim \pi_2^{\#}(N\,;\,n) \sim \pi_2(N)\text{, as } n \to \infty \qquad (24)$$

Clearly, it follows from (8) that $F_N(\theta) > 0$ for all θ. This, of course, implies that $J_N(\theta) > 0$ for all θ. We then see that the summands in (19) and (20) are positive ; therefore the estimates in (19) and (20) must also be positive. In point of fact, we are only moderately interested in such estimates ; these are provided only as numerical checks in the subsequent computations. Our real interest lies in the estimates given by (22) and (23).

If n in (22) is replaced by 2n, we obtain, after simplification :

$$\pi_2^*(N;2n) = \pi_2^\#(N;n) \qquad (25)$$

We then see that the two estimates in (22) and (23) are essentially equivalent. Henceforth, we therefore use only the estimate in (23), and omit the superscript notation "#" .

For computational purposes, we have used N = 1000, n = 500 ; after some rather extensive computations, we find that $\pi_2(1000; 500) = 36.54$. Also, we find that $\pi^*(1000; 500) = 181.76$. The true values for N = 1000 are as follows : $\pi^*(1000) = 167$, and $\pi_2(1000) = 35$.

The computations imply that the estimate in (23) is *apparently* dominated by the value of $G_N(0)$. This observation, of course, needs to be confirmed. The goal of the development in the following section is to reach this conclusion, which will imply TPC.

4. ESTIMATE OF G_N(θ AND PROOF OF TPC :

Our starting point is the following generalization of the Prime Number Theorem (PNT) :

$$\sum_{p \le N} p^a \sim \frac{N^{a+1}}{(a+1)\log N}, \ a = 0, 1, 2, \ldots \qquad (26)$$

This may be obtained by partial integration of the auxiliary function $\int_2^N \frac{x^a}{\log x} dx$; the derivation of (26) is left as an exercise for the reader. It might be noted that this appears in SIREV, in slightly different form [2]. Note that for a = 0, (26) yields PNT. The asymptotic behavior is, of course, exhibited as $N \to \infty$. We may also remark that the sum in (26) is not substantially changed, in an asymptotic sense, if we exclude the prime p = 2 and restrict p to be odd.

In this section, we use the relation in (26) to derive certain relations that evaluate the quantities $F_N(\theta)$ and $G_N(\theta)$ for all (positive) values of θ. For our purpose, we restrict θ for now to the interval [0, π/4]. Make the following definitions :

$$C_N(\theta) \equiv \sum_{3\le p\le N} \cos p\theta \ , \ S_N(\theta) \equiv \sum_{3\le p\le N} \sin p\theta \tag{27}$$

As we recall, by (8) :

$$F_N(\theta) = \{C_N(\theta)\}^2 + \{S_N(\theta)\}^2 \tag{28}$$

Using (27) : $C_N(\theta) = \sum_{3\le p\le N} \sum_{k=0}^{\infty} (-1)^k \frac{(p\theta)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} \sum_{3\le p\le N} p^{2k}$ **. Now using (26), we obtain :**

$C_N(\theta) \sim \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k}}{(2k)!} \left\{ \frac{N^{2k+1}}{(2k+1)\log N} \right\}$ **, or :**

$$C_N(\theta) \sim \frac{\sin(N\theta)}{\theta \log N} \tag{29}$$

Also, $S_N(\theta) = \sum_{3\le p\le N} \sum_{k=0}^{\infty} (-1)^k \frac{(p\theta)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \sum_{3\le p\le N} p^{2k+1} \sim \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \left\{ \frac{N^{2k+2}}{(2k+2)\log N} \right\}$,

or :

$$S_N(\theta) \sim \frac{1-\cos(N\theta)}{\theta \log N} \tag{30}$$

It then follows from (28) that

$$F_N(\theta) \sim \frac{4 \sin^2(N\theta/2)}{\theta^2 \log^2 N} \tag{31}$$

We now make the relatively innocuous assumption that N is a multiple of 4 , or N = 4M, say.

With this assumption, $\sin\{N(\pi/2 - \theta)/2\} = \sin\{M\pi - N\theta/2\} = (-1)^{M-1} \sin(N\theta/2)$ **; then**

$$F_N(\pi/2 - \theta) \sim \frac{4 \sin^2(N\theta/2)}{\{\pi/2 - \theta\}^2 \log^2 N} \tag{32}$$

From the definition given in (17), it follows that if $\theta \neq 0$, $\theta \neq \pi/2$:

$$G_N(\theta) \sim, \frac{4\,\sin^2(N\theta/2)}{\log^2 N}\left\{\frac{1}{\theta^2} - \frac{1}{(\pi/2-\theta)^2}\right\} \qquad (33)$$

It is not difficult to show that $G_N(0) = 4\,\pi_{4,1}(N)\,\pi_{4,3}(N)$, where $\pi_{4,j}(N) = |\{p \le N : p \equiv j \pmod 4\}|$, with j = 1 or 3 . We know that the sets $\pi_{4,1}(N)$ and $\pi_{4,3}(N)$ are *equinumerous*, meaning that $\pi_{4,1}(N) \sim \pi_{4,3}(N)$ as $N \to \infty$. This implies that each such quantity is $\sim \frac{1}{2}\pi^*(N)$ as $N \to \infty$.

Therefore, we see that $G_N(0) \sim \{\pi^*(N)\}^2$. The first positive value of θ that occurs as the argument of $G_N(\theta)$ in formula (23) is $\theta = \pi/4n$. We let $\theta = k\pi/4n$, for $1 \le k \le n$, the values that appear as positive arguments of G_N in (23). Making this substitution in (33), this yields :

$$G_N(k\pi/4n) \sim \frac{64n^2\ \sin^2(Nk\pi/8n)}{(k\pi)^2\log^2 N}\left\{1 - \frac{k^2}{(2n-k)^2}\right\}$$ **, or equivalently, as $N \to \infty$:**

$$G_N(k\pi/4n) \sim \left\{\frac{N}{\log N}\right\}^2 \left\{\frac{\sin(Nk\pi/8n)}{Nk\pi/8n}\right\}^2 \left\{1 - \frac{k^2}{(2n-k)^2}\right\}, \ (k = 1, 2, \cdots, n) \qquad (34)$$

We note that over the interval [0, n] , the quantity $\left\{1 - \frac{k^2}{(2n-k)^2}\right\}$ is a decreasing function of k with range [0, 1] . As we already know, $F_N(0) = \{\pi^*(N)\}^2 \sim \frac{N^2}{\log^2 N}$, by PNT. In the form given by (34), it is clear that if we let $k \to 0$, $G_N(0) \sim \left\{\frac{N}{\log N}\right\}^2 \sim \{\pi^*(N)\}^2 = F_N(0)$, once again.

Now the asymptotic formula given in (34) is valid as $N \to \infty$, but says nothing about asymptotic behavior as $n \to \infty$: However, if we are to apply the formula in (23), we must consider n to be increasing indefinitely, as well as N . We may get around this difficulty by supposing that n and N increase indefinitely in a dependent manner ; we find it convenient to suppose that N = 2n. Henceforth, asymptotic behavior will implicitly be described as $N \to \infty$. In our computations, we assumed that N = 1000, n = 500. As a further refinement in our computations, we may replace the quantity $\left\{\frac{N}{\log N}\right\}^2$ by $\{\pi^*(N)\}^2$, which is certainly valid asymptotically. We then make the definition :

$$G_N^{\#}(k\pi/2N) = \{\pi^*(N)\}^2 \left\{\frac{\sin\{k\pi/4\}}{k\pi/4}\right\}^2 \left\{1 - \frac{k^2}{(N-k)^2}\right\}, \; k = 1, 2, \cdots, N/2 \qquad (35)$$

We may also define $G_N^{\#}(0) = \{\pi^*(N)\}^2$, the limiting case in (35) as $k \to 0$. We may also extend the domain of k to k = 1+N/2 , 2+N/2, $\cdots$, N – 1 , and make the additional definition for k = N : $G_N^{\#}(\pi/2) = -\{\pi^*(N)\}^2$.

As we have seen (from (15), (23) and our other assumptions):

$\pi_2(N) \sim \frac{1}{N}\sum_{k=0}^{N/2} \cos(k\pi/N)\, G_N(k\pi/2N)$. We may also assert that $G_N(k\pi/2N) \sim G_N^{\#}(k\pi/2N)$.

It then follows that :

$$\pi_2(N) \sim \frac{1}{N}\sum_{k=0}^{N/2} \cos(k\pi/N)\, G_N^{\#}(k\pi/2N) \qquad (36)$$

The advantage of using $G_N^{\#}(k\pi/2N)$ in (36), as opposed to $G_N(k\pi/2N)$, is that $G_N^{\#}(k\pi/2N)$ is non-negative over $k \in [0, N/2)$; clearly, $G_N^{\#}(k\pi/2N) = 0$ iff k = 4, 8, $\cdots$, 4[M/2] . The same may not be said of $G_N(k\pi/2N)$, as verified by computation. Although incidental here, we may show from the definitions that the G_N's and $G_N^{\#}$' s share some interesting properties. Since a description of these properties would be distracting to the narrative, we indicate these in the Appendix.

Returning to the expression in (36), its other component, namely cos(kπ/N), is of course positive and decreasing over $k \in [0, N/2)$ (and vanishes for k = N/2) . We may therefore assert the following consequence of (36) :

$$\pi_2(N) > \frac{G_N^{\#}(0)}{N} \qquad (37)$$

Since $G_N(0) \sim \left\{\frac{N}{\log N}\right\}^2$, it follows that as $N \to \infty$:

$$\pi_2(N) > \frac{N}{\log^2 N} \qquad (38)$$

Since $\frac{N}{\log^2 N}$ is an unbounded function of N, this implies TPC!

On the other hand, the following known result is attributable to Brun :

$$\pi_2(N) = O\left(\frac{N}{\log^2 N}\right) \tag{39}$$

Together with (38), this further shows that

$$\pi_2(N) \propto \frac{N}{\log^2 N} \tag{40}$$

5. THE STRONG TWIN PRIME CONJECTURE :

The famous Hardy-Littlewood Conjecture given in [1] (also known as the "strong" Twin Prime conjecture, which we denote as STPC) is an extension of TPC, and provides an estimate for the value of $\pi_2(N)$, which we now know is an unbounded function of N :

$$\pi_2(N) \sim 2C_2 \frac{N}{\log^2 N} \sim 2C_2 \int_3^N \frac{dx}{\log^2 x} \tag{41}$$

Here, C_2 (the so-called Twin Primes Constant) is given by $C_2 = \prod_{p \geq 3}\left(1 - \frac{1}{(p-1)^2}\right)$, where the product is over all odd primes. Approximately, $C_2 \approx 0.6601618158$. No attempt will be made in this paper to prove STPC.

6. A MORE GENERAL CONJECTURE :

It would appear that the method indicated in this paper may be used similarly (with certain modifications), to prove a variety of more general number-theoretic conjectures. However, no rigorous proofs are given in this section or subsequent sections ; we merely indicate a skeleton of a proof of how a rigorous development for the more general cases might proceed. The most obvious generalization that comes to mind is the conjecture that $\pi_{2k}(N)$ is unbounded ; here, $\pi_{2k}(N)$ is the counting function of the prime pairs (p, p+2k) with $p \leq N$. Our starting point for this is the more general function :

$$g_{2k:N}(z) = \sum_{n=3}^{N} \delta(n)\delta(n+2k)\, z^n \quad , N = 3, 4, \ldots ; \quad k=1, 2, \ldots \qquad (42)$$

Incidentally, this explains the use of the notation $g_{2:N}(x)$ used in the case of the twin primes. The first non-zero exponent appearing in the expansion of (42) need not necessarily be equal to 3 ; for example, if k = 3, such exponent is 5, since the first such prime pair of the type (p, p+6) is (5, 11) and not (3, 9).

If the counting function of the prime pairs (p, p+2k) is denoted as $\pi_{2k}(N)$, note that $\pi_{2k}(N) = g_{2k:N}(1)$. A development similar to the preceding one yields the following :

$$\pi_{2k}(N) = \frac{1}{2\pi}\int_0^{2\pi} \cos 2k\theta \, | f_N(e^{i\theta}) |^2 \, d\theta \qquad (43)$$

By the same process as used previously for the twin primes, we find that

$$\pi_{2k}(N) = \frac{2}{\pi}\int_0^{\pi/2} \cos 2k\theta \, | f_N(e^{i\theta}) |^2 \, d\theta \qquad (44)$$

As before, we may deduce that $\pi_{2k}(N)$ is an unbounded function of N.

7. <u>PRIME TRIPLETS</u> :

Another application that comes to mind involves the function $\pi_{2,4}(N)$, which counts the number of prime triplets (p, p+2, p+6) with $p \le N$. In order to prove that $\pi_{2,4}(N)$ is unbounded, we would begin with the polynomial function :

$$g_{2,4:N}(z) = \sum_{n=3}^{N} \delta(n)\delta(n+2)\delta(n+6)\, z^n \quad , N = 3, 4, \ldots \qquad (45)$$

In a more general case, we would define $g_{2u,2v:N}(x) = \sum_{n=3}^{N} \delta(n)\delta(n+2u)\delta(n+2u+2v)\, x^n$. Note that, in order to avoid trivial cases (e.g., u = v = 1), we would need to impose additional conditions on u and v, in order to ensure that none of the triplet components are divisible by 3 (except possibly 3 itself) ; we find that we must exclude the cases where

u ≡ v ≡ ±1 (mod 3). Assuming that such conditions are in place, we make the following definition :

$$\pi_{2u,2v}(N) = \sum_{n=3}^{N} \delta(n)\,\delta(n+2u)\,\delta(n+2u+2v) \;=\; g_{2u,2v:N}(1) \qquad \textbf{(46)}$$

We proceed by a process similar to the development indicated in Section 2 .

Then $\pi_{2u,2v}(N) = \sum_{n=3}^{N} \delta(n)\,\delta(n+2u)\,\frac{1}{2i\pi}\oint_{\Gamma} \frac{f_N(z)}{z^{n+2u+2v+1}}\,dz \;=\; \frac{1}{2i\pi}\oint_{\Gamma} \frac{f_N(z)}{z^{2u+2v+1}} \sum_{n=3}^{N} \delta(n)\,\delta(n+2u)\left(\frac{1}{z}\right)^{n} dz$ **,**

or by reference to the definition in (44) :

$$\pi_{2u,2v}(N) = \frac{1}{2i\pi}\oint_{\Gamma} \frac{f_N(z)\,g_{2u:N}(1/z)}{z^{2u+2v+1}}\,dz \qquad \textbf{(47)}$$

After further simplification, we obtain :

$$\pi_{2u,\,2v}(N) = \frac{1}{2\pi}\mathrm{Re}\int_{0}^{2\pi} e^{-2i(u+v)\theta}\, f_N(e^{i\theta})\, g_{2u:N}(e^{-i\theta})\,d\theta \;. \qquad \textbf{(48)}$$

It may be shown that the expression in (50) has the order $\frac{N}{\log^3 N}$**, which in turn shows that** $\pi_{2u,\,2v}(N)$ **is unbounded. Clearly, this process can be extended to deal with any m-tuples of primes that are to be spaced apart non-trivially at pre-determined intervals.**

8. SOPHIE GERMAIN PRIME PAIRS :

We may also deal in like manner with the Sophie Germain primes, another class of prime pairs that has not been discussed in the preceding treatment. The Sophie Germain primes, named in honor of the early 19th century French female mathematician, are those primes p such that 2p+1 is also prime. The initial functions that we use to deal with these are $g_{SG:N}(z) = \sum_{n=3}^{N} \delta(n)\,\delta(2n+1)\; z^{n}$ **, and** $\pi_{SG}(N) = \sum_{n=3}^{N} \delta(n)\,\delta(2n+1) \;=\; g_{SG:N}(1)$ **. The proof that there exist infinitely many Sophie Germain pairs (p, 2p+1) follows by a process that is analogous to that previously used. We omit the derivation, and simply indicate the result :**

$$\pi_{SG}(N) = \frac{1}{2\pi}\mathrm{Re}\int_{0}^{2\pi} e^{-i\theta}\; f_N(e^{i\theta})\,f_N(e^{-2i\theta})\;d\theta \qquad \textbf{(49)}$$

The integrand reduces to the following : $\sum \cos p\theta \sum\; \cos(2p+1)\theta + \sum \sin\theta \sum\; \sin(2p+1)\theta$ **;**

in this expression, all sums are over odd primes $p \le N$.

Again using the estimates previously derived in (29) and (30) for $C_N(\theta)$ and $S_N(\theta)$, we obtain the following estimates :

$$\sum \cos(2p+1)\theta = \cos\theta\, C_N(2\theta) - \sin\theta\, S_N(2\theta) \sim \frac{\cos\theta \sin 2N\theta - \sin\theta\,(1-\cos 2N\theta)}{2\theta \log N}, \text{ or}$$

$$\sum \cos(2p+1)\theta \sim \frac{\cos(N+1)\theta \sin N\theta}{\theta \log N} \qquad (50)$$

Also, $$\sum \sin(2p+1)\theta = \cos\theta\, S_N(2\theta) + \sin\theta\, C_N(2\theta) \sim \frac{\sin\theta \sin 2N\theta + \cos\theta\,(1-\cos 2N\theta)}{2\theta \log N}, \text{ or}$$

$$\sum \sin(2p+1)\theta \sim \frac{\sin(N+1)\theta \sin N\theta}{\theta \log N} \qquad (51)$$

Then, after simplification :

$$\sum \cos p\theta \sum \cos(2p+1)\theta + \sum \sin\theta \sum \sin(2p+1)\theta$$

$$\sim \frac{2 \sin N\theta \sin(N\theta/2) \cos((1+N/2)\theta)}{\theta^2 \log^2 N} . \qquad (52)$$

We see that this has the same order of magnitude as $F_N(\theta)$; by the same reasoning as applied to the case of the twin primes, it follows that the integral in (49) is unbounded.

In this case, as well as in certain more general situations involving m-tuples of primes, the applicable expressions appear to lend themselves to comparable analysis.

No doubt, many other interesting examples will occur to the reader. Some of these, at least, appear to be susceptible to the method indicated in this paper (with suitable modifications) . In all such cases, the applicable counting function is shown to be unbounded from the unboundedness of the corresponding integral. Generally, such m-tuples of primes will be of order $\frac{N}{\log^m N}$, which is an unbounded function of N .

APPENDIX

The following properties are easily proved by the appropriate definitions, over the domain

k = 0, 1, …, N :

$$G_N^{\#}(k\pi/2N) = -G_N^{\#}(\pi/2 - k\pi/2N)\ ;\ G_N(k\pi/2N) = -G_N(\pi/2 - k\pi/2N)\ ; \qquad \text{(i)}$$

Equivalently, if k is replaced by N – k , the negative of the original term is obtained ; special considerations must be made at the end points k = 0 and k = N . The relations in (i) easily imply the following :

$$\sum_{k=0}^{N} G_N^{\#}(k\pi/2N) = 0\ ;\ \sum_{k=0}^{N} G_N(k\pi/2N) = 0\ ; \qquad \text{(ii)}$$

$$G_N^{\#}(\pi/4) = 0\ ;\ G_N(\pi/4) = 0\ ; \qquad \text{(iii)}$$

Also, the following easily follows from the definition of G_N in (17) :

$$\int_0^{\pi/2} G_N(\theta)\, d\theta = 0\ , \qquad \text{(iv)}$$

Another interesting result is the following :

$$\frac{1}{N}\sum_{k=0}^{N/2} G_N^{\#}(k\pi/2N) \sim \frac{2.5N}{\log^2 N} \qquad \text{(v)}$$

Proof of (v) : From the definition of $G_N^{\#}$ given in (35), the special value at k = 0, and PNT, we see that the sum in (v) (call it S_N) is ~ $\frac{N}{\log^2 N}\left(\frac{16}{\pi^2}\right)\{\frac{\pi^2}{16} + \sum_{k=0}^{N/2}\left(\frac{\sin^2(k\pi/4)}{k^2}\right) - \sum_{k=N/2}^{N}\left(\frac{\sin^2(k\pi/4)}{k^2}\right)\}$ **.**

As $N \to \infty$, the second sum in the preceding expression tends to vanish. If J_N represents the sum $\sum_{1 \le k \le N/4} \frac{1}{(2k-1)^2}$ **, we then see that $S_N \sim$** $\frac{N}{\log^2 N}\left(\frac{16}{\pi^2}\right)\left\{\frac{\pi^2}{16} + \left(\frac{1}{2}\right)J_N + \left(\frac{1}{4}\right)J_N\right\}$**. Also, it is easily shown that $J_N \to \frac{\pi^2}{8}$ as $N \to \infty$. Then $S_N \sim$** $\frac{N}{\log^2 N}\left(\frac{5}{2}\right)$ **, after simplification, which proves (v) .**

In view of the last result, the formula given in (36) and STPC (assuming the latter to be true), we may regard the "average cosine" appearing in (36) to equal $2C_2/(2.5) = (0.8)C_2$, which is approximately equal to 0.52813. By this computation, such "average θ" appearing in (36) is roughly 58°.

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