A PROOF OF THE RIEMANN HYPOTHESIS

By

PAUL S. BRUCKMAN
38 Front Street, #302
Nanaimo, BC V9R 0B8 (Canada)
Phone and Fax : (250) 591-2290
e-mail : pbruckman@hotmail.com

ABSTRACT :

An elementary proof of the Riemann Hypothesis (RH) is presented. We begin with the known Laurent expansion for $\zeta(1\text{-}s)$, where $s \neq 0$ is a complex number in the critical strip and ζ is the Riemann Zeta function. This implies a simple power series identity satisfied by any non-trivial zero ρ. In turn, this implies another pair of power series in terms of $P = \rho(1 - \rho)$ and $\overline{P}$. Under the assumption that ρ is not on the critical line, the last two power series may be subtracted, yielding yet another power series in terms of $|P|$. We show, however, that the latter power series is ill-defined, which implies that $P = \overline{P}$, i.e., ρ must lie on the critical line, proving RH.

KEY WORDS OR TERMS

Riemann Hypothesis ; Riemann Zeta function ; non-trivial zeros ; critical strip ; critical line ; Stieltjes numbers ; open critical quarter-strip

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1. INTRODUCTION :

An elementary proof of the famous Riemann Hypothesis (RH) is reported. A moderate knowledge of the theory of the Riemann Zeta function is assumed throughout this paper. An excellent exposition of many aspects of this theory is given by Derbyshire in his book [1]. Also, the known formulas related to the Riemann Zeta function that are indicated in this paper are given in [2]. Appropriate tables of values are found at the web sites indicated in [3] and [4]. Following standard practice, $\zeta(s)$ is the Riemann Zeta function, where $s = x + it$ is any complex number $\neq 1$; ζ is analytic everywhere except at $s = 1$, where it has a simple pole with a residue equal to 1 . We let $\tilde{S}$ denote the (closed) critical strip $\{s = x+it : 0 \leq x \leq 1, -\infty < t < \infty\}$ and L the critical line $\{s = 1/2+it : -\infty < t < \infty\}$; also, S denotes the open critical strip $\{s = x+it : 0 < x < 1, -\infty < t < \infty\}$.

We let $\rho = \sigma + i\tau$ denote any non-trivial zero of ζ (i.e. lying in S). As we know, $\zeta(\rho) = \zeta(1 - \rho) = \zeta(\overline{\rho}) = \zeta(1 - \overline{\rho}) = 0$. Due to the symmetry of the zeros, we will find it convenient to define the "open critical quarter-strip" $S_1 = \{s = x+it : 0 < x < 1/2, t > 0\}$. Also, it is known that there are no zeros of ζ on either of the external boundaries of S or on the semi-axis segment $0 \leq x \leq 1$.

Because of this, in order to prove RH, it suffices to prove that there are no non-trivial zeros of ζ in S_1 . In fact, all the *known* non-trivial zeros of ζ lie on L (and occur in conjugate pairs) . Riemann himself hypothesized in 1859 that all the non-trivial zeros of ζ lie on L . G. H. Hardy, in 1914, proved that there are infinitely many zeros of ζ on L . Given that the non-trivial zeros of ζ above the x-axis are ordered by magnitude of modulus, Xavier Gourdon and Patrick Demichel, in 2004, showed that the first 10^{13} zeros all lie on L . We may label such zeros as ρ_1, ρ_2, ρ_3, etc. If there are any $\rho \notin L$, such ρ must satisfy $|\rho| > 2.5 * 10^{12}$, approximately. It may be added that the first three non-trivial zeros (with $\tau > 0$), are (approximately) $\rho_1 \approx 1/2 + 14.134725 14i$, $\rho_2 \approx 1/2 + 21.02203964i$ and $\rho_3 \approx 1/2 + 25.01085758i$; then $|\rho_1| \approx 14.14356585$, $|\rho_2| \approx 21.02798494$, $|\rho_3| \approx 25.01585491$, also approximately .

2. ADDITIONAL DEVELOPMENT :

We begin with the following known Laurent expansion for $\zeta(1 - s)$, valid in S :

$$\zeta(1 - s) = \frac{-1}{s} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} s^n , \qquad (1)$$

where the γ_n's are the *Stieltjes numbers*, given by :

$$\gamma_n = \lim_{N \to \infty} \left\{ \sum_{k=1}^{N} \frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1} \right\} \qquad (2)$$

In particular, setting s = ρ, we obtain the expansion :

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \rho^{n+1} = 1 \qquad (3)$$

Note that (3) is valid for all ρ, and that the γ_n's are independent of any ρ ; it should also be noted that the approximate values of the γ_n's are known to great accuracy, up to at least $n \leq 78$ (see [3]) , and vary in sign, with no discernible pattern in their

sign being apparent .

Given the convergence of the series in (3), it follows that $\frac{|\gamma_n|}{n!} = o\left(\frac{1}{|\rho|^{n+1}}\right)$.

We note that the γ_n's are independent of any value of ρ . We write $r = |\rho|$, $r_1 = |\rho_1|$, etc. Since the ρ's are any non-trivial zeros of ζ , we may say that

$$\frac{|\gamma_n|}{n!} = \left(\frac{a_n}{r_1}\right)^{n+1}, \ n = 0, 1, 2, \ldots, \text{ where } a_n \geq 0 \, , \ a_n = o(1) \tag{4}$$

Since the relation in (3) is valid for all $\rho \in S$, it is also true for $1 - \rho$.

Note that $\rho \neq 1 - \rho$, i.e., $\rho \neq 1/2$; also, $\rho \neq 0$, $1 - \rho \neq 0$. We may therefore subtract the two relations (after first dividing by ρ and $1 - \rho$, respectively), as follows :

$\sum_{n=1}^{\infty} \frac{\gamma_n}{n!}\left\{\rho^n - (1-\rho)^n\right\} = \frac{1}{\rho} - \frac{1}{1-\rho} = \frac{1-2\rho}{\rho(1-\rho)}$. For brevity, write $P = \rho(1 - \rho)$. We also employ the following well-known identity, valid for all x and y and natural n :

$$x^n - y^n = (x - y) \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} (x+y)^{n-1-2k} (-xy)^k \tag{5}$$

If we set $x = \rho$ and $y = 1 - \rho$ in (5), we may simplify the foregoing equation as follows :

$\frac{1-2\rho}{P} = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!}(2\rho - 1) \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} (-P)^k$. We may cancel the term $(1 - 2\rho)$, since $\rho \neq 1/2$.

We then obtain : $\sum_{k=0}^{\infty} (-P)^{k+1} \sum_{n=0}^{\infty} \binom{n+k}{n} \frac{\gamma_{n+2k+1}}{(n+2k+1)!} = 1$, or by a change of notation :

$$\sum_{n=0}^{\infty} U_n (-P)^{n+1} = 1 \, , \tag{6}$$

where

$$U_n = \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{\gamma_{k+2n+1}}{(k+2n+1)!} \tag{7}$$

Let $Q \equiv |P|$. Note that the U_n's are independent of the ρ's, hence of P or Q .

As in the previous argument, it follows that $|U_n| = o\left(\frac{1}{Q^{n+1}}\right)$ for all Q .

Computations have been performed of the quantities U_n for $0 \le n \le 100$, and these are shown in Table 1 . Also, the sum indicated in (6) has been computed for P_k, k = 1, 2, 3, and has been found to converge to 1 within the limits of the accuracy attainable with the computational program . Naturally, the number of terms required to attain a pre-determined degree of accuracy in the sum increases as P_k increases.

We find that $P = \sigma(1-\sigma) + \tau^2 + i\tau(1-2\sigma)$; hence $Q^2 = P\overline{P} = \{\sigma(1-\sigma) + \tau^2\}^2 + \tau^2(1-2\sigma)^2$.

Let $R = \mathrm{Re}(P) = \sigma(1-\sigma) + \tau^2$. Note that P = Q = R iff $\rho \in L$; otherwise, Q > R .

Also note that $P_1 = (\tau_1)^2 + ¼ \approx 200.040451$; $P_2 = (\tau_2)^2 + ¼ \approx 442.1761506$;

$P_3 = (\tau_3)^2 + ¼ \approx 625.7929969$.

2. PROOF OF RH :

At this point, we assume that $\rho \in S_1$, which implies that $\mathrm{Im}(P) = \tau(1-2\sigma) > 0$. Under this assumption, it is clear that $P \neq \overline{P}$. Returning to (6), if we replace ρ by $\overline{\rho}$, P is replaced by $\overline{P}$, hence $\sum_{n=0}^{\infty} U_n\left(-\overline{P}\right)^{n+1} = 1$. Subtracting this from equation (6) as before, we obtain : $\sum_{n=1}^{\infty} U_n(-1)^{n+1}\left\{P^n - \overline{P}^n\right\} = \frac{1}{P} - \frac{1}{\overline{P}} = \frac{-(P-\overline{P})}{P\overline{P}} = \frac{-2i\,\mathrm{Im}(P)}{Q^2}$. On the other hand, using the identity in (5) again, this time with x = P, $y = \overline{P}$, we obtain the following :

$$\frac{-2i\,\mathrm{Im}(P)}{Q^2} = \sum_{n=1}^{\infty}(-1)^{n+1}U_n\{2i\,\mathrm{Im}(P)\}\sum_{k=0}^{[(n-1)/2]}\binom{n-1-k}{k}\{2R\}^{n-1-2k}\{-Q^2\}^k$$

$$= \{2i\,\mathrm{Im}(P)\}\sum_{k=0}^{\infty}(-1)^k Q^{2k}\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{n}\{2R\}^n U_{n+2k+1}$$, or after a change in notation :

$$\{2i\,\mathrm{Im}(P)\} = \{2i\,\mathrm{Im}(P)\}\sum_{n=0}^{\infty}(-1)^{n+1}Q^{2n+2}V_n \, , \qquad (8)$$

where

$$V_n = \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} \{2R\}^k U_{k+2n+1} \tag{9}$$

Although the U's in this last expression are independent of the ρ's, this is not true of the R's and of the V's ; therefore, a more appropriate notation for V_n would be $V_n(\rho)$. Cancelling the factor 2i Im(P) (= 2iτ(1 - 2σ)) from both sides of (8), as allowed under the assumption that $\rho \in S_1$, we obtain :

$$\sum_{n=0}^{\infty} \{-Q^2\}^{n+1} V_n = 1 \tag{10}$$

Arguing as before, we deduce that $| V_n | = o\left\{\frac{1}{Q^{2n+2}}\right\}$, hence $V_n \to 0$ as $n \to \infty$.

On the other hand, since the U's in (9) are independent of the ρ's and R's, the absolute value of each term of the sum in (9) increases without bound as r increases, due to the presence of the term $(2R\}^k$. Therefore, the sum "defining" V_n in (9) is divergent.

This contradicts the result that $V_n \to 0$ as $n \to \infty$; thus, the definition of V_n is somehow faulty. Our fallacy in defining V_n was our assumption that $\rho \in S_1$, which is equivalent to the assumption $P \neq \overline{P}$; this, in turn, is equivalent to assuming that $\sigma \neq 1/2$. We conclude, therefore, that we must have $\sigma = 1/2$, thus proving RH !

3. CONCLUSION AND ACKNOWLEDGMENTS :

The author wishes to dedicate this paper posthumously to his former professor and mentor A.O.L. Atkin, Emeritus of the University of Illinois. Also, the author acknowledges the significant contributions made by Mr. Larry Shultis, who provided extensive computations using Mathematica, corroborating some of the assertions made in this paper. Finally, I would be remiss if I omitted thanking Dr. John C. Turner, Emeritus of the University of Waikato, for furnishing me with a copy of Derbyshire's excellent book [1] and for invaluable aid in formatting this paper for proper presentation,

as well as his insertion of this paper on his web site [5] . Grateful acknowledgment is also made to the anonymous referee(s) for valuable suggestions that tended to improve this presentation .

The author is well aware of the profound effect that this proof will have on many aspects of number theory. In particular, numerous results that have been stated as being conditional on the truth of the Riemann Hypothesis may now be laid to rest as bona fide theorems. It is not the intention of this author to enumerate such previously "conditional" results ; this has been done elsewhere and much more comprehensively by numerous mathematicians of superior quality. We will only mention a pair of such results that struck this author as particularly interesting, and that may now be stated as unconditional theorems ; the first deals with the gaps between consecutive primes, the second with the difference between the prime counting function and the logarithmic integral :

(*) **If $\{p_n\}$ is the sequence of primes, then $(p_{n+1} - p_n) \leq (p_n)^{1/2} \log p_n$** **(11)**

(**) $$\pi(x) = Li(x) + O(x^{1/2} \log x) \qquad (12)$$

Undoubtedly, many more previously conditional results will occur to the reader.

TABLE 1

n	U_n	n	U_n	n	U_n
0	-0.07721566	41	4.396E-113	82	-1.5199E-253
1	0.00055402	42	-8.856E-117	83	-4.9704E-257
2	4.9011E-06	43	-2.238E-119	84	-6.0173E-261
3	-1.3911E-07	44	-1.044E-122	85	9.7808E-265
4	3.0123E-10	45	1.352E-126	86	5.9882E-268
5	8.6073E-12	46	4.105E-129	87	1.1889E-271
6	-2.8038E-14	47	1.933E-132	88	3.4171E-276
7	-2.8319E-16	48	-8.149E-137	89	-4.7589E-279
8	7.2411E-19	49	-5.594E-139	90	-1.4461E-282
9	6.0233E-21	50	-2.782E-142	91	-1.7232E-286
10	-7.2257E-24	51	-1.23E-146	92	1.7488E-290
11	-8.3114E-26	52	5.639E-149	93	1.1797E-293
12	-7.9128E-30	53	3.093E-152	94	2.3097E-297
13	7.0516E-31	54	3.787E-156	95	1.1758E-301
14	8.4084E-34	55	-4.123E-159	96	-5.9206E-305
15	-3.328E-36	56	-2.641E-162	97	-1.8955E-308
16	-8.1497E-39	57	-5.077E-166	98	-2.4731E-312
17	5.5601E-42	58	2.059E-169	99	6.9902E-317
18	3.7728E-44	59	1.718E-172	100	1.0060E-319
19	2.3185E-47	60	4.431E-176		
20	-8.6221E-50	61	-5.568E-180		
21	-1.5279E-52	62	-8.408E-183		
22	3.9544E-56	63	-2.756E-186		
23	3.5569E-58	64	-7.199E-191		
24	2.824E-61	65	3.003E-193		
25	-3.1501E-64	66	1.262E-196		
26	-7.2218E-67	67	1.584E-200		
27	-2.4227E-70	68	-7.223E-204		
28	6.6986E-73	69	-4.287E-207		
29	8.6093E-76	70	-8.756E-211		
30	5.2497E-80	71	8.104E-215		
31	-7.6324E-82	72	1.066E-217		
32	-6.6678E-85	73	3.028E-221		
33	7.5233E-89	74	1.647E-225		
34	5.4536E-91	75	-1.835E-228		
35	3.6082E-94	76	-7.358E-232		
36	-7.5427E-98	77	-1.067E-235		
37	-2.636E-100	78	1.746E-239		
38	-1.44E-103	79	1.279E-242		
39	3.383E-107	80	2.856E-246		
40	9.007E-110	81	7.24E-251		

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