ON THE OCCURRENCE OF PRIMES WITHIN VARIOUS SEQUENCES

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ABSTRACT : The purpose of this paper is to present a heuristic argument that there exist infinitely many primes within certain common sequences of natural numbers ; in particular, we argue that there exist infinitely many (a) Fibonacci primes ; (b) Lucas primes ; (c) Mersenne primes ; (d) prime repunits ; (e) Pell primes, and (f) Pell-Lucas primes. A generalization of these conjectures that encompasses these sequences is also presented. It is also argued that there exist infinitely many composite elements of these sequences. Furthermore, we indicate a general conjecture regarding the precise magnitude of the counting function of the primes occurring within these various sequences ; we also compare these conjectures with the latest known values of these counting functions. Also, we argue that there exist only finitely many Fermat primes.

KEY WORDS OR TERMS : Fibonacci primes ; Lucas primes ; Mersenne primes ; prime repunits ; Pell primes ; Pell-Lucas primes ; Fermat primes ; counting functions ; asymptotic behavior.

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1. INTRODUCTION :

Among the many unsolved problems of number theory are the questions of whether there exist infinitely many primes among (a) the Fibonacci numbers ; (b) the Lucas numbers ; (c) the Mersenne numbers ; (d) the repunits ; (e) the Pell numbers ; and (f) the Pell-Lucas numbers . In this paper, we present a heuristic argument in favor of positive answers to these questions ; in other words, we argue (but do not prove) that the six sequences named do contain infinitely many primes (as has been previously conjectured). In addition, we argue that there are only finitely many Fermat primes. The argument employed has general application to a wide variety of linear sequences.

2. THE SEQUENCES STUDIED :

For purposes of review, the Fibonacci numbers and Lucas numbers both satisfy the common recurrence relation :

$$G_{n+2} = G_{n+1} + G_n \,,\; n = 0, 1, 2, \ldots . \qquad (1)$$

The initial conditions for the Fibonacci numbers are : $G_0 = 0$, $G_1 = 1$; for the Lucas numbers, the initial conditions are : $G_0 = 2$, $G_1 = 1$. The Fibonacci sequence is commonly denoted as $\{F_n\}$ and the Lucas sequence as $\{L_n\}$. Explicit expressions (known as Binet expressions) for the Fibonacci and Lucas numbers are given by :

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \,,\; L_n = \alpha^n + \beta^n \,,\; n = 0, 1, 2, \ldots, \text{ where } \alpha = \frac{1+\sqrt{5}}{2},\; \beta = \frac{1-\sqrt{5}}{2}. \qquad (2)$$

In order for F_n to be prime, n must be prime, except for n = 4 ; however, the converse is false ; for example, F_{19} = 4,181 = 37 * 113. As of this writing, the subscripts of the known (or suspected) Fibonacci primes are as follows : n = 3, 4, 5, 7, 11, 13, 17, 23, 29, 43, 47, 83, 131, 137, 359, 431, 433, 449, 509, 569, 571, 2971, 4723, 5387, 9311, 9677, 14431, 25561, 30757, 35999, 37511, 50833, 81839, <u>104911</u>, <u>130021</u>, <u>148091</u>, <u>201107</u>, <u>397379</u>, <u>433781</u>, <u>590041</u>, <u>593689</u>, <u>604711</u> . The underlined items above correspond to *probable* prime Fibonacci numbers . These indices (42 in all) have been obtained from the Wolfram web site [1] .

A similar situation exists for the Lucas primes. A Lucas number is prime only if n = 0, n is prime or a power of 2 . The converse is false : for example, $L_3 = 4 = 2^2$, and L_{32} = 4,870,847 = 1,097 * 4,481 . The corresponding indices for known (or suspected) Lucas primes are as follows : n = 0, 2, 4, 5, 7, 8, 11, 13, 16, 17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353, 503, 613, 617, 863, 1097, 1361, 4787, 4793, 5851, 7741, 8467, 10691, 12251, 13963, 14449, 19469, 35449, 36779, 44507, 51169, 56003, <u>81671</u>, <u>89849</u>, <u>94823</u>, <u>140057</u>, <u>148091</u>, <u>159521</u>, <u>183089</u>, <u>193201</u>, <u>202667</u>, <u>344293</u>, <u>387433</u>,

443609, 532277, 574219, ... Again, underlined items above correspond to *probable* Lucas primes. These indices (57 in all) have been obtained from the Wolfram web site [2] . Note that 5 of these indices are 0 or powers of 2 .

The Mersenne numbers $M_p \equiv 2^p - 1$ can only be prime if p is prime, though the converse is false ; for example, M_{11} = 2,047 = 23 * 89. For our purposes, we will find it convenient to define the Mersenne numbers M_n for *all* integers n, even though they are normally defined only for n = p, a prime. The primes p for which M_p is known or suspected to be prime, are as follows : p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 43112609 ; as before, underlined entries correspond to probable primes.

Indeed, the largest known primes are Mersenne primes. The effort to expand the foregoing list is part of the Great Internet Mersenne Prime Search (GIMPS), a computer project in which non-professional and amateur mathematicians are invited to participate. The Electronic Frontier Foundation (EFF) offers a prize of $100,000 USD to the individual or group who first discovers a prime with 10,000,000 decimal digits! EFF also offers a prize of $150,000 for discovery of the first 10^8-digit prime, and $250,000 for discovery of the first 10^9-digit prime! The foregoing indices (44 in all) have been obtained from the Wolfram web site [3] .

The repunits $R_n \equiv (10^n - 1)/9$ are the strings of n 1's in the decimal system . Again, in order for R_n to be prime, n must be prime, although the converse is false. As of this writing, the known prime (or probable prime) repunits occur for n = 2, 19, 23, 317, 1031, <u>49081</u>, <u>86453</u>, <u>109297</u>, <u>270343</u> (9 in total) . As before, underlined items correspond to probable primes, according to [4] .

The Pell and Pell-Lucas numbers satisfy the common recurrence relation :

$$H_{n+2} = 2H_{n+1} + H_n \, , \; n = 0, 1, 2, \ldots. \qquad (3)$$

The initial conditions for the Pell numbers are : $H_0 = 0$, $H_1 = 1$; for the Pell-Lucas numbers, the initial conditions are : $H_0 = 1$, $H_1 = 1$. The Pell sequence is commonly denoted as $\{P_n\}$ and the Pell-Lucas sequence as $\{Q_n\}$. Explicit formulas for these numbers are given below :

$$P_n = \frac{u^n - v^n}{u - v} \, , \; Q_n = (u^n + v^n)/2 \, , \; n = 0, 1, 2, \ldots, \text{ where } u = 1 + \sqrt{2} \, , \; v = 1 - \sqrt{2} \, . \qquad (4)$$

In some sources, the Pell-Lucas numbers are defined as 2 times the numbers as defined here. P_n is prime only if n is prime, while Q_n is prime only if n is prime or a power of 2. The known (or suspected) indices of P_n such that P_n is prime are as follows : 2, 3, 5, 11, 13, 29, 41, 53, 59, 89, 97, 101, 167, 181, 191, 523, 929, 1217, 1301, 1361, 2087, 2273, 2393, 8093, 13339, <u>14033</u>, <u>23747</u>, <u>28183</u>, <u>34429</u>, <u>36749</u>, <u>90197</u> ; for Q_n , the corresponding indices are as follows : 2, 3, 4, 5, 7, 8, 16, 19, 29, 47, 59, 163, 257, 421, 937, 947, 1493, 1901, 6689, 8087, 9679, <u>28753</u>, <u>79043</u>, <u>129127</u>, <u>145969</u>, <u>165799</u>, <u>168677</u>, <u>170413</u>, <u>172243</u> . As before, underlined items correspond to probable primes. The foregoing indices (31 for P_n and 29 for Q_n) have been obtained from the Wolfram web site [5] .

The Fermat numbers are numbers of the form $2^{2^n}+1$, n = 0, 1, 2, … . In 1650, Fermat conjectured that all such numbers are prime. Euler, in 1732, found that $2^{2^5}+1$ is composite, with a factor of 641, thus disproving Fermat's conjecture. Eisenstein, in 1844, proposed that there might be infinitely many Fermat primes. In fact, the only 5 Fermat numbers known to be prime are for n = 0, 1, 2, 3, and 4, namely : 3, 5, 17, 257 and 65,537, as stated in [6]. As we will argue in this paper, there appear to be only finitely many Fermat primes. Moreover, we will also indicate some probability that there is possibly one other prime further in this sequence.

3. ADDITIONAL DEVELOPMENT :

The first six of the sequences introduced in the previous section are of the general form $A_n \sim c\,\theta^n$ as $n \to \infty$, for positive constants c and $\theta > 1$; it is also assumed that $A = \{A_n\}_{n\geq 1}$ is a non-decreasing infinite sequence of positive integers . For simplicity, we also assume that $A_1 = 1$. The six sequences initially studied in this paper satisfy these criteria, and moreover, are second-order linear sequences ; however, there is nothing to prevent linear sequences of order other than two in the generalizations.

There is another important consideration that needs to be addressed, namely the question of irreducibility. Given a general sequence of the form indicated, we must impose the additional condition that A_n is irreducible in integers. For example, if $A_n = (4^n - 1)/3$, this can be factored as follows : $(2^n + 1)(2^n - 1)/3$, which (except for a few initial values of n) will necessarily be a composite integer. A somewhat subtler example of an "intrinsically composite" sequence is the following : $A_n = (4^{4n+3} + 1)/5 = (2^{4n+3} + 2^{2n+2} + 1)(2^{4n+3} - 2^{2n+2} + 1)/5$; note that one or the other factor is divisible by 5, hence A_n is a composite integer (except for n = 0). These examples were suggested by Carl Pomerance in private correspondence with one of the authors (Bruckman).

The Fibonacci, Mersenne, repunit and Pell sequences are of the general form $(x^n - y^n)/(x - y) = x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}$, which is irreducible in integers for $n \geq 3$. The Lucas and Pell-Lucas sequences are of the general form $(x^n + y^n)/(x + y)$, which for odd $n \geq 3$ is equal to $x^{n-1} - x^{n-2}y + \ldots - xy^{n-2} + y^{n-1}$, also irreducible in integers ; for all n, these sequences produce integral values. In any event, the six sequences under study have forms that are irreducible in the integers. It must therefore be a necessary condition that the results of this paper can only apply to forms that are irreducible in integers. An open question is whether this condition is also sufficient for applicability. Namely, the question arises : do all sequences that have forms that are irreducible in integers contain infinitely many primes?

In the case of the Fibonacci numbers, $c = \frac{1}{\sqrt{5}}$ and $\theta = \alpha \equiv \frac{1+\sqrt{5}}{2}$; for Lucas numbers, $c = 1$, $\theta = \alpha$; for Mersenne numbers, $c = 1$, $\theta = 2$; for repunits, $c = \frac{1}{9}$, $\theta = 10$; for Pell numbers, $c = \frac{1}{\sqrt{2}}$, $\theta = u = 1 + \sqrt{2}$; and for Pell-Lucas numbers, $c = \frac{1}{2}$, $\theta = u$.

Fermat numbers are examples of a second class of sequences, of the general form : $B_n \sim \exp(A_n)$ as $n \to \infty$, where A_n is of the first class ; in the case of Fermat numbers, $c = \log 2$, $\theta = 2$ for the corresponding sequence $\{A_n\}_{n\geq 1}$.

Given a sequence $A = \{A_n\}_{n\geq 1}$ of the first class, let $\delta_A(n)$ denote its characteristic

function, and make the following definition :

$$\pi_A(N) = \sum_{n=1}^{N} \delta_A(n) = \sum_{A_n \le N} 1 \qquad (5)$$

This is the counting function of the elements A_n of A with $A_n \le N$. Similarly, let $\delta_\Pi(n)$ denote the characteristic function of the prime sequence. We then define the counting function $\pi_{\Pi A}(N)$ as follows :

$$\pi_{\Pi A}(N) = \sum_{n=1}^{N} \delta_\Pi(n)\,\delta_A(n) = \sum_{p \le N,\ p \in A} 1 \qquad (6)$$

This is the counting function of the *prime* elements A_n of A with $A_n \le N$.

If the subscript "A" is replaced by "F", "L", "M" , "R", "P" or "Q", so as to denote applicability to the specific sequences $F = \{F_n\}_{n\ge1}$, $L = \{L_n\}_{n\ge1}$, $M = \{M_n\}_{n\ge1}$, $R = \{R_n\}_{n\ge1}$, $P = \{P_n\}_{n\ge1}$ or $Q = \{Q_n\}_{n\ge1}$, respectively, we obtain after some numerical verification :

$$\delta_F(n) = \left[\frac{\log\{1+n\sqrt{5}\}}{\lambda}\right] - \left[\frac{\log\{1+(n-1)\sqrt{5}\}}{\lambda}\right] - \delta_{n:1}, \text{ where } \lambda = \log\alpha , \qquad (7)$$

for $n \ge 2$; also, $\delta_F(1) = 1$. The "$\delta_{n:1}$" is the Kronecker delta. Note that since $F_1 = F_2 = 1$, we only count n = 1 once in the definition of $\delta_F(1)$;

$$\delta_L(n) = \left[\frac{\log\{n-\beta\}}{\lambda}\right] - \left[\frac{\log\{n-1-\beta\}}{\lambda}\right] - \delta_{n:2} + \delta_{n:3} ; \qquad (8)$$

Note that although $L_0 = 2$, we consider $2 \notin L = \{L_n\}_{n\ge1}$, hence $\delta_L(2) = 0$;

$$\delta_M(n) = \left[\frac{\log\{1+n\}}{\log 2}\right] - \left[\frac{\log n}{\log 2}\right] ; \qquad (9)$$

$$\delta_R(n) = \left[\frac{\log\{1+9n\}}{\log 10}\right] - \left[\frac{\log\{-8+9n\}}{\log 10}\right] ; \qquad (10)$$

$$\delta_P(n) = \left[\frac{\log\{1+2n\sqrt{2}\}}{\xi}\right] - \left[\frac{\log\{1+2(n-1)\sqrt{2}\}}{\xi}\right], \text{ where } \xi = \log u ; \qquad (11)$$

$$\delta_Q(n) = \left[\frac{\log\{2n+1\}}{\xi}\right] - \left[\frac{\log\{2n-1\}}{\xi}\right] . \qquad (12)$$

The formulas given in (7)-(12) are easily verified by replacing n with A_n and, as expected, yield $\delta_A(n) = 1$ if $n \in A$, and $\delta_A(n) = 0$ if $n \notin A$.

These characteristic functions (except for the Kronecker delta function "adjustments") are of the form $[a_n] - [a_{n-1}]$. Applying the definition in (5), we see that this is a telescoping sum (except for one or two terms), and obtain : $\pi_A(N) = [a_N] - [a_0] - \Delta = [a_n] - \Delta$, since $[a_0] = 0$ in all cases ; here, Δ is an "adjustment" due to the appearance of any Kronecker delta functions (such as appear in (7) and (8)). The next step is to remove the brackets ; clearly, $\pi_A(N) \sim a_N$. We see that the formulas given in (7)-(12) produce $a_N \sim \log(cN)/\log\theta$ for some sequence-specific constant c (respectively, $c = \sqrt{5}$, 1, 1, 9, $2\sqrt{2}$ and 2) . Therefore, $\pi_A(N) \sim a_N \sim \log N/\log\theta$. This implies that positive constants $J = J_A$ and $K = K_A$ exist such that for all N :

$$J \log N/\log\theta < \pi_A(N) < K \log N/\log\theta \qquad (13)$$

This relation has been verified numerically, up to certain large values of N, for the sequences F, L, M, R, P and Q ; the minimum constant J appears to be approached at relatively small values of N .

Perhaps a better way to verify (13) is to note that $\pi_A(A_M) = M$; hence, substituting $N = A_M \sim c\theta^M$, we see that $M \sim \dfrac{\log(N/c)}{\log\theta}$. Hence $\pi_A(N) \sim \log(N/c)/\log\theta \sim \log N/\log\theta$ as $N \to \infty$, which is equivalent to (13) .

Up to now, our argument has been precise and has not involved any heuristics. Now, however, we delve into the realm of what might be characterized as "trial and error". It is important to note that, in what follows, hardly anything is proved ; the arguments employed, however plausible, are not rigorous proofs.

We observe that since $\pi_A(N) = \sum_{n=1}^{N} \delta_A(n) \sim \log N/\log\theta$, and $\sum_{n=1}^{N} \frac{1}{n} \sim \log N$, the "average magnitude" of $\delta_A(n)$ is $\frac{1}{n\log\theta}$. We may also arrive at this conclusion less precisely, by applying the principle that first differences are approximately equal to first derivatives ; this, of course, applies only to differentiable functions, in accordance with the Mean Value Theorem of calculus. Specifically, we first remove the brackets from the bracket functions $[a_n] - [a_{n-1}]$ in (7)-12). Next, we replace $a_n - a_{n-1}$ with a'_n , treating n temporarily as a continuous variable. For example, doing this in (7) yields the approximate relation : $\delta_F(n) \approx \frac{\sqrt{5}}{(1+n\sqrt{5})\lambda} \sim \frac{1}{n\lambda}$. More generally, we conclude that the "average magnitude" of $\delta_A(n)$ is $\frac{1}{n\log\theta}$. This implies that in any sum that involves $\delta_A(n)$, we may approximate such sum, in some sense, by replacing $\delta_A(n)$ with $\frac{1}{n\log\theta}$. Of course, such approximation for $\delta_A(n)$ cannot be accurate for any specific n, since in fact, $\delta_A(n) = 0$ or 1 ; we speak only of "average" behavior , with sums involving an aggregation of, rather than individual, $\delta_A(n)$'s.

4. ARGUMENT THAT $\pi_{\Pi A}(N)$ IS UNBOUNDED :

We return to the definition of $\pi_{\Pi A}(N)$ given in (6) . Since A is an infinite sequence, we argue that the substitution of $\frac{1}{n\log\theta}$ for $\delta_A(n)$ may be made throughout the sum in that definition, provided we are concerned only with the order of magnitude of the sum . Accordingly, we argue that $\pi_{\Pi A}(N) \propto \sum_{n=1}^{N} \frac{\delta_\Pi(n)}{n\log\theta} = \frac{1}{\log\theta}\sum_{p\le N}\frac{1}{p} \sim \frac{1}{\log\theta}\log\log N$, using a well-known number-theoretic result [7] . In other words, we argue that :

$$\pi_{\Pi A}(N) \propto \log\log N \tag{14}$$

It seems clear from (14) that $\pi_{\Pi A}(N)$ is an unbounded function of N . If we could accept our argument at face value, this would show that there are infinitely many

primes occurring in A . In particular, we would be able to argue that there are infinitely many prime elements of F, L, M, R, P and Q .

5. <u>ARGUMENT THAT THERE ARE FINITELY MANY FERMAT PRIMES</u> :

In this section, we will argue that the number of Fermat primes is finite. The Fermat numbers are normally denoted as F_n in the literature ; however, since that notation was used previously to denote the Fibonacci numbers, we will use $\phi = \{\phi_n\}_{n\geq 0}$ to denote the sequence of Fermat numbers, where $\phi_n = 2^{2^n} + 1$. Recall that ϕ is an example of a "second-class" sequence $B = \{B_n\}_{n\geq 0}$, where $B_n \sim \exp(A_n)$, and $A_n \in A$, a "first-class" sequence. In this case, if $\phi_m = n$, then

$$m = \frac{\log\log(n-1) - \log\log 2}{\log 2}, \text{ and so if } n \geq 3,$$

$$\delta_\phi(n) = \left[\frac{\log\log(n-1) - \log\log 2}{\log 2}\right] - \left[\frac{\log\log(n-2) - \log\log 2}{\log 2}\right] \text{; also, } \delta_\phi(1) = \delta_\phi(2) = 0 \,.$$

The "average order" of $\delta_\phi(n)$ (in the sense previously discussed) is found by noting that $d(\log\log x)/dx = 1/(x \log x)$; hence, we deduce that the average order of $\delta_\phi(n)$ is $1/(n \log n)\log 2$. Therefore, our argument implies that

$$\pi_{\Pi\,\phi}(N) \propto \sum_{n=1}^{N} \frac{\delta_\Pi(n)}{n \log n} = \sum_{p\leq N} \frac{1}{p \log p}.$$

We now take the time to prove that the infinite series $\sum_p \frac{1}{p \log p}$ converges to a limit, which implies that the sum $\sum_{p\leq N} \frac{1}{p \log p}$ is bounded by such limit ; this, in turn (accepting our argument), would imply that $\pi_{\Pi\phi}(N)$ is a constant for all sufficiently large N, i.e., there are finitely many Fermat primes.

<u>THEOREM</u> : The series $\sum_p \frac{1}{p \log p}$, summed over all primes p, converges .

<u>Proof</u> : The Prime Number Theorem implies that the n-th prime p_n satisfies :

$p_n \sim n \log n$ as $n \to \infty$; hence, $\log p_n \sim \log n$, and $p_n \log p_n \sim n \log^2 n$.

Therefore, $\sum_p \frac{1}{p\log p} \propto \sum_{n=2}^{\infty} \frac{1}{n\log^2 n} \propto \int_2^{\infty} \frac{dx}{x\log^2 x}$. However, this last integral is exactly integrable as $\frac{-1}{\log x}\Big]_2^{\infty} = \frac{1}{\log 2}$, a constant. We therefore conclude that the value of the series $\sum_p \frac{1}{p\log p}$ is a constant. □

As argued above, this shows that there exist finitely many Fermat numbers .

Incidentally, we may verify numerically that the (constant) value of $\sum_p \frac{1}{p\log p}$ is approximately equal to 1.566 (which, in turn, is approximately equal to $\frac{1}{\log(1.894)}$) .

6. <u>CONJECTURES AND APPROXIMATIONS</u> :

We have argued that the order of magnitude of $\pi_{\Pi A}(n)$ is log log n . The following more precise general conjecture may be made :

$$\pi_{\Pi A}(n) \sim \frac{e^{\gamma}\ \log\log n}{\log\theta} \quad , \qquad (15)^*$$

where γ is the Euler Constant. For example, Crandall and Pomerance make this conjecture in [8], with regard to the Mersenne primes (where $\theta = 2$). The same kind of heuristic argument should apply to the other sequences, thereby justifying (15)*. We may test this conjecture (denoted by an asterisk) for specific sequences .

For example, (15)* would imply that $\pi_{\Pi F}(n) \sim \frac{e^{\gamma}\ \log\log n}{\log\alpha}$ and also that $\pi_{\Pi L}(n) \sim \frac{e^{\gamma}\ \log\log n}{\log\alpha}$. If we assume that the probable primes in Section 2 are indeed prime, we obtain : $\pi_{\Pi F}(F_{604711}) = 42$; by comparison, $\frac{e^{\gamma}\ \log\log(F_{604711})}{\log\alpha} \approx 46.6$. Also, we obtain : $\pi_{\Pi L}(L_{574219}) = 57$, while $\frac{e^{\gamma}\ \log\log(L_{574219})}{\log\alpha} \approx 46.4$. If the conjecture in (15)* should prove to be valid, this would indicate that some of the probable

Lucas primes shown in Section 2 are likely to actually be composite.

For the Mersenne sequence, we obtain : $\pi_{\Pi M}(M_{32582657}) = 44$, while

$\frac{e^{\gamma} \log\log(M_{32582657})}{\log 2} \approx 43.51$. The corresponding statistics for repunits are as follows : $\pi_{\Pi R}(R_{270343}) = 9$, while $\frac{e^{\gamma} \log\log(R_{2703437})}{\log 10} \approx 10.3$. For Pell numbers, we obtain : $\pi_{\Pi P}(P_{90197}) = 31$, while $\frac{e^{\gamma} \log\log(P_{90197})}{\log u} \approx 22.8$; and for Pell-Lucas numbers, $\pi_{\Pi Q}(Q_{172243}) = 29$, while $\frac{e^{\gamma} \log\log(Q_{172243})}{\log u} \approx 24.1$. Both statistics suggest that some of the "probable primes" may in fact be composite.

A corresponding conjecture may be made for "B-type" sequences of the second class :

$$\pi_{\Pi B}(n) \approx \frac{e^{\gamma} S}{\log \theta} \quad \text{for all sufficiently large } n \text{, where } S = \sum_{p} \frac{1}{p \log p} \approx 1.566 \qquad (16)^*$$

In this case, the "≈" notation might be loosely interpreted as "very near". For the case of the Fermat Primes, $\pi_{\Pi\phi}(\phi_4) = \pi_{\Pi\phi}(65537) = 5$, while $\frac{e^{\gamma} S}{\log 2} \approx 4.02$; no primes in this sequence are known beyond $\phi_4 = 65537$; however, the data seems to indicate that there may possibly be one more prime ϕ_n for some $n \geq 5$.

No attempt is made in this paper to prove or disprove the conjectures made in (15)* and (16)* , nor even to explain the motivation for making them ; again, see [8] for a "how to" argument for making such conjectures.

7. COMPOSITE NUMBERS IN THE SEQUENCES :

It has been conjectured that there are infinitely many *composite* Mersenne numbers . We will generalize this conjecture. Let $S_{\Pi A}(N) = \{A_p : p \leq N, p \text{ prime}\}$,

and $Q_{\Pi A}(N) = |S_{\Pi A}(N)|$. Also, let $\pi_{CA}(N)$ denote the counting function of the composite elements of $\{A_n\}$. According to our argument, $\pi_{\Pi A}(N) \propto \log\log N$ $= o(\pi(N)) = o\left(\frac{N}{\log N}\right)$, since $\pi(N) \sim \frac{N}{\log N}$, while $\pi_{CA}(N) = \pi(N) - \pi_{\Pi A}(N) \sim \frac{N}{\log N}$, an unbounded function. This argues for the infinite number of composite elements of the sequence $\{A_n\}$. Incidentally, this is clearly also true for sequences that are "reducible" in integers.

8. <u>CONCLUSION</u> :

As previously indicated, the conjecture indicated in (15)* applies to many other general sequences of first or higher order . The argument employed, with certain modifications, remains essentially plausible for other linear sequences. No doubt, numerous other applications will occur to the reader.

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MATHEMATICS SUBJECT CLASSIFICATIONS : 11B39, 11B83, 11N05