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FIFTH DRAFT

(0,1)-PATTERNS GENERATED BY A DOUBLE-CYCLING PROCESS; WITH A PROOF OF THE TWIN PRIMES CONJECTURE

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This paper is a FOURTH draft only. There is much that still needs editing. We are hopeful that not only will a much shorter paper evolve, but also a quicker method of proof of the twin primes conjecture will be discovered on the way. Tantalizing glimpses of one keep arising as we go on mulling the many facts that have been discovered so far.

John Turner, 20th September, 2011

Fifth Draft ... Jan 2012

<u>Note:</u> A different proof of the TPC emerged on New Year's Eve, 2011
It is hoped to present this alternative proof at the Fibonacci Association Biennial Conference, in Hungary, June 2012.

(0,1)-PATTERNS GENERATED BY A DOUBLE-CYCLING PROCESS; AND A PROOF OF THE TWIN PRIMES CONJECTURE

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ABSTRACT

The main purpose of this paper is to settle the well-known twin primes conjecture, which we claim to be true.

In order to make good our claim, we create an alternative model of the natural numbers, in terms of certain (0,1)-sequences, which are produced sequentially within a doubly-infinite two-way matrix ***C****. We study the (0,1)-patterns in the rows and columns of the matrix, and develop methods for deciding when the 'new-model primes' and the 'new-model twin primes' occur in the matrix, in such detail that we are able to justify our claim..*

God created the integers, All the rest is the work of Man..
L. Kronecker (1823—1891) [6]
God gave us neither zero nor the rules of the numbers game
-- Giuseppe Peano made that Good.
G. Peano (1858—1932) [5], J. C. Turner (2010)

PREAMBLE

We are going to take the unusual step in a mathematics paper of starting with a synopsis of the methods which lead to a proof of the main theorem, viz. the twin primes conjecture. We are doing this because there are several major points about the paper which we wish to stress at the outset.

First we point out that the methods we use are completely new to us, so we can expect the reader will have initial difficulty in grasping certain concepts which we introduce, and appreciating the mathematical language in which they are written. In fact, they are all simple enough, but there is a welter of small ideas and consequences which have to be defined, explained and exemplified, before their use in the final proof can be understood and accepted as providing valid proof.

The development of the infrastructure, as it were, begins by defining an alternative model of the natural numbers. This is built within a doubly infinite matrix **C**, whose elements all belong to {0,1}. From then on, the methods and tools to be used are all constructed from certain (0,1)-patterns within **C,** and many relations and properties of these special patterns are discovered and examined. Theorems about these patterns are given in Part I, and some of the results from them are related back to similar (often the 'same' in a dual sense) results about the natural numbers.

In order to specialize these methods so that the twin primes can be singled out, again as (0,1)-patterns, a second (0,1)-matrix is constructed, derived from **C** by making use of Boolean additions of pairs of elements from rows of **C**. We denote this matrix by **C′**, and find the required twin prime (0,1)-patterns within **C′.**

The final point we wish to stress in this preamble, is that as the reader wades through the often verbose, even repetitive, materials given in the next 40 or so pages, and studies the examples we give, *we ask him/her to remember throughout that it is (0,1)-patterns being studied, not the natural numbers themselves.* It is through the direct correspondences between the rows, say r_j of **C,** and the integers j, that we come to assert that (0,1)-patterns within and between the rows and columns tell us something about the numbers themselves.

The main example of patterns that we can mention now, is one which turns out to be the crux in the proof of existence of infinities of both primes and twin primes. It is that of sequences of 2x1 vectors $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\end{pmatrix}$, (2-vectors for short), which we designate by *f* and *g* respectively. They occur infinitely often within the last two rows of certain cycling Block matrices (which are derived from **C** or **T**), when their Partial Boolean Row Products are computed. We can infer from occurrences of the 2-vector *f* s when prime- or twin-prime columns are being formed, or when potential twin prime columns cease to be potential twin primes (pTPs). And from occurrences of *g* s we can infer when pPs and pTPs continue through sieves, maintaining the possibility that they will eventually become Ps or TPs.

We believe that we might have left readers of the last paragraph behind. They may no longer understand what we are telling them. Well, that just confirms the point made in the second paragraph above. A lengthy infrastructure must be built, concepts absorbed, language terms and phrases assimilated, before understanding can take place. As an analogous example, before Euclid gave his demonstration of the infinity of primes, he expected his readers to have read up to Proposition 19 in book IX of his Elements of Geometry (c. 300 B.C.) [6]; a lengthy preparation indeed.

1. INTRODUCTION

As stated in the Abstract, the aim of this paper is to prove the twin-primes conjecture, which we shall henceforth refer to by the acronym TPC. A twin-prime is a pair of consecutive odd numbers in the sequence of natural numbers, both of which are prime. Sometimes it is helpful to relate the pair to a triple of consecutive integers, say *(n-1, n, n+1),* where both *n-1* and *n+1* are prime numbers. We shall call *n-1* the left-arm (or L-arm) of the twin-prime, and *n+1* the right-arm (or R-arm) of the twin-prime; we may refer to *n* as the mid-term.

We shall begin by developing an alternative model of the natural numbers, whereby we can study numbers in new ways, in terms of infinite (0,1)-strings in a matrix **C.** Methods for finding the primes, using properties of columns of **C,** will be discovered. Then another matrix **C'** will be derived from **C**, which will enable us to discover the twin-primes sequentially, in a similar manner to that used with **C** for primes**.**

A counting algorithm will be presented, which will enable us to show that neither the primes, nor the twin-primes, can ever come to an end. That is, both the Primes Conjecture (PC) and the Twin-Primes Conjecture (TPC) are true.

PART I --- THE MATRIX C : PROOF OF THE PRIMES CONJECTURE

In this Part we provide the definitions, symbols and language needed to present a new model of the natural numbers, and to give a new proof of the Primes Conjecture, (acronym PC). We construct and use a (0,1)-matrix **C** which contains the new numbers, and discover many of its properties. The new numbers are called *cycle-numbers.*

The presentation in this Part is descriptive and somewhat verbose, with many examples and figures. In Part II, we derive the second (0,1)-matrix **T,** from **C**, and use it to prove the Twin-Primes Conjecture (the TPC). The proof is virtually a dual of that for the PC, so the presentation is less verbose and moves at a faster pace.

1.1 The cycle-number and its fundamental cycle (f.c.)

Definition 1.1: A *cycle-number* $\underline{\mathbf{n}}$ is a potentially infinite (0,1)-sequence which is generated by a *fundamental cycle* of (0,1) elements, of length n, and denoted by $\underline{\mathbf{n}}^{(1)}$. We often abbreviate 'fundamental cycle' to f.c. The fundamental cycle is repeated indefinitely (either element-wise, in single steps, or in entire cycles) to form the cycle-number $\underline{\mathbf{n}}$.

The construction of the fundamental cycle is explained below, in 2.2 and defn. 3.2.

N.B.: It may be that a fundamental cycle is itself generatable by a (0,1)-vector of length $d<n$ and $d|n$. We call such a vector a *base cycle* (e.g. (1,0) generates $\underline{\mathbf{4}}$, whose fundamental cycle is (1,0,1,0), see Figure 2.1.

Definition 1.2: **C** is *the doubly-infinite matrix* whose *n*th row is $\underline{\mathbf{n}}$.

Before we explain how we derive the fundamental cycles, it will be instructive to show the first five of them, and indicate how we first derived them.

Later we shall describe a double-cycling process which produces them sequentially within rows of the 2-way matrix **C**.

2. THE FIRST FIVE FUNDAMENTAL CYCLES

*Figure 2.1. The first five fundamental cycles in **C** derived from n-words*

Integer n	n-word	Fundamental cycle $\underline{n}^{(1)}$
1	(1_1)	(1)
2	$(2_1, 2_2)$	(1,0)
3	$(3_1, 3_2, 3_3)$	(1,1,0)
4	$(4_1, 4_2, 4_3, 4_4)$	(1,0,1,0)
5	$(5_1, 5_2, 5_3, 5_4, 5_5)$	(1,1,1,1,0)

Figure 2.1 shows the natural numbers 1 to 5 in column 1, then a column of so-called n-words, and finally a column of the promised first five fundamental cycles.

It is evident that there is a one-to-one correspondence between any two of these columns, on the grounds that the lengths of the vectors in columns 2 and 3 are equal to n, and clearly there are no repetitions in each downward sequence.

The appearances of 1s and 0s in the third column are explained in subsection 2.2 below.

The construction of the n-words is self-evident, although it requires further comment. An n-word is a vector consisting of the number n repeated n times, with subscripts 1 to n attached in natural order. In fact, the lower numbers are not mere subscripts, as position indicators, but integers in their own right. Microsoft Word has not enabled us to write them with the same font size as that of the upper integers. A brief explanation of this matter will now be given.

2.1 Brief history of the entegers

The first-named author (with A. G. Schaake) introduced the concept of *enteger*, in a paper called '*The Elements of Enteger Geometry*' [4,1993]. The domain of the geometry was a binary tree, whose nodes were assigned ordered pairs of integers. Originally (in Schaake's Regular Knot Tree [2]) each integer was a parameter value for a given type of braid. Schaake would write the two parameters p and b on a node thus: p/b. It did not mean a rational number, nor did it mean a pair (p,b); and not a 2-dimensional vector.

p meant the number of parts, and b meant the number of bights, respectively for a particular type of braid which Schaake was studying. Each number had its own significance.

After Turner had worked with Schaake on his braid theories for a while in the late 80s, he decided to omit the tiresome forward-slash between the p and the b, when drawing up similar evolutionary trees (as so they were) and write simply p_b , with both letters having the same size. Later Turner considered the possibility of developing a number theory for these entities. He decided to give them the name '*enteger*', suggesting an entity part way between integer and rational number. A few definitions of terms about entegers will now be given.

Definitions 2.1:

(i) An '*enteger e'* is an ordered pair of integers, say s,t , the first being the *'upper'* and the second the *'lower'*, written as $e \equiv s_t$.

(ii) s is called the '*stem*' of the enteger, and t the '*tail*'.

(iii) An '*enteger word'* $\underline{e}$ of length n is a vector of n entegers.

(iv) The *'coprime operator (kappa)*' is defined as follows:

$$\kappa(e) \equiv \begin{cases} 1 & if \;\; \gcd(s,t) = 1 \\ 0 & if \;\; \gcd(s,t) > 1 \end{cases}$$

N.B. In this paper there is no mention of divisibility, nor of factors of numbers. None is necessary. We could replace the notion of 'greatest common divisor' (i.e. gcd) in the definition of *kappa* by the notion 'least positive difference' (i.e. lpd) and use only repeated subtractions to carry out Euclid's Algorithm, in order to find values of lpd(s,t). Euclid's Algorithm is the foundational tool of modern number theory, on which divisibility theory is built. We have said at the end of our paper that 'coprimeness begets primeness and not vice versa': thus we shall define in 2.3 the primeness of a cycle-number in terms of the 'coprimeness' of its fundamental cycle. This paper demonstrates that our concept of 'coprimeness' is a powerful one for use in studying relations between the natural numbers.

Thinking about possibilities for formulating number theories of entegers led Turner to study the *enteger n-words,* and thence to the *cycle-numbers* which initiated the tools developed in this paper. He felt it in order to present this brief history of entegers, since they pointed the way to our TPC proof given below. Apart from the next paragraph they will rarely appear again. (We could in fact avoid using them altogether, and work with matrix row and column subscripts alone, as is explained later.)

An alternative derivation of the fundamental cycles from matrices will be dealt with below, in 2.3, for they will provide many of the tools that we use to prove the TPC. There we shall use only the alphabet {0,1} and derive a sequence of matrices ($\mathbf{C}_n$) ; natural numbers will not enter this derivation, except for their use as ordering subscripts. In a later paper we shall give yet another way of arriving at the cycle-numbers, when we construct a (0,1)-triangle by a method analagous to that for producing Pascal's Arithmetical Triangle (1665).

We next show how the nth fundamental cycle (f.c.) is derived from the n-word.

2.2 Computation of the fundamental cycles (see def. 1.1) from n-words

Applying the kappa operator to each of the elements of an n-word, we obtain the fundamental cycle $\underline{\mathbf{n}}^{(1)}$. Thus, taking n = 5 for example:

$$\kappa(5_1, 5_2, \cdots, 5_5) = (\kappa(5_1), \kappa(5_2), \cdots, \kappa(5_5)) = (1,1,1,1,0) = \underline{\mathbf{5}}^{(1)} .$$

We see that in the general case the elements of $\underline{\mathbf{n}}^{(1)}$ are gcd(n,i) for $i = 1, 2, \cdots, n$.

Four useful propositions about $\underline{\mathbf{n}}^{(1)}$ are stated within the following theorem:

Theorems 2.1:

(i) The first n-1 elements of $n^{(1)}$ form a palindrome (the Palindrome Law).
Proof: gcd(n, i) = gcd(n, n-i) for i = *1* , ... , *n-1*.

(ii) If n is odd, the central term of $\underline{\mathbf{n}}^{(1)}$ is 1 .
Proof: $\gcd(n, \lceil n/2 \rceil) = 1$ (note the 'ceiling brackets').

(iii) For all n > 1, the first and penultimate elements of $\underline{\mathbf{n}}^{(1)}$ are 1s, and the last term is 0.

Proof: Evident from the definitions and kappa values.

(iv) Let the weight ω of a fundamental cycle be the count of 1s in its vector. Then: $\omega(\underline{\mathbf{n}}^{(1)}) = \varphi(n)$, where φ is Euler's totient function.

Proof: Follows immediately from definitions of ω and φ.

2.3 Observations and definition of primes:

We see immediately from Figure 2.1 a pattern in the vectors for $\underline{\mathbf{n}}^{(1)}$ when n is prime. For <u>2</u>, <u>3</u>, and <u>5</u> we observe (1,0), (110), and (11110) respectively, which suggests that we make the following definition:

Definition 2.2: (see definition 1.1)

(i) $\underline{\mathbf{n}}^{(1)}$ is a *prime fundamental cycle* iff its *(0,1)*-pattern is (1,1,1, ... ,1,0) , i.e. (*n-1)* 1*s* followed by 0.

(ii) $\underline{\mathbf{n}}$ is a *prime cycle-number* iff its fundamental cycle is prime.

We shall generally use *p* to signify a prime, and $\underline{\mathbf{p}}$ to signify its corresponding cycle-number. Then using theorem 2.1(iv) we find $\omega(\underline{\mathbf{p}}^{(1)}) = \varphi(p) = p - 1$.

The last element of a fundamental cycle is on the leading diagonal (l.d.) of **C,** and is therefore 0 (= $\kappa(n_n)$). Therefore the fundamental cycle of $\underline{\mathbf{p}}$ is prime, so $\underline{\mathbf{p}}$ is a prime cycle-number, by def, 2.2(i).

3. FUNDAMENTAL CYCLES FROM DOUBLE-CYCLING MATRICES

Before presenting further properties of the fundamental cycles, we must show how they can be obtained from a sequence of (0,1) matrices. Note carefully that in Section 2, we appealed directly to the natural numbers in order to introduce certain names and properties and ideas. In this section we shall use only 0s and 1s, and when natural numbers occur, they will only act as ordering subscripts.

3.1 Construction method for the matrix sequence $\{\mathbf{C_n}\}$

The matrices in the sequence are square, with all elements being 0 or 1. Their sequence begins with a 1x1 matrix $\mathbf{C_1}$ whose single element is 1 ; then a single-column and single-row gnomon is added to form $\mathbf{C_2}$, a 2x2 matrix ; another gnomon is added to that, to form a 3x3 matrix $\mathbf{C_3}$; and so on, creating a potentially endless sequence of ever expanding square matrices. The question of what is a gnomon, and how are its (0,1)-elements determined, must be addressed. A description of how a gnomon is to be constructed will now be given. The reader may refer to the sequence of five examples which are given below the definition, for clarification.

(i) Definition and construction of a gnomon

The gnomon to be added to $\mathbf{C_n}$ is an 'angle bracket' with a 'lower arm' and a 'right arm', the arms being of equal length *n*, and a 'corner' which provides the *(n+1)*th diagonal element of $\mathbf{C_{(n+1)}}$.

The right arm is filled by one-step cycling of the fundamental cycles in the rows of $\mathbf{C_n}$, to the right until the right arm is filled in column *n+1*. Similarly the lower arm is filled by one-step cycling of the fundamental cycles in the columns of $\mathbf{C_n}$ down to the lower arm in row $n+1$. The element in r_{n+1} and c_{n+1} is set to 0.

By decree, the corner element in $\mathbf{C_i}$,after $\mathbf{C_1}$**,** is always to be set to 0.

As n tend to infinity, the matrices tend to a doubly infinite matrix **C.**

Definition 3.1: We shall call **C** the *'coprimeness matrix of the integers'*, or else, with equal validity, *'the cycle-number matrix'.*

Five examples of the double-cycling method of matrix constructions follow:

Examples: Construction of $\{\mathbf{C_n}\}$ for $n = 1 \text{ to } 5$

Explanations and comments about the gnomons follow Figure 2.2 below:

$$\mathbf{C}_1 = \begin{bmatrix} 1 \end{bmatrix} \text{, the given initial matrix.}$$

$$\mathbf{C}_2 = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & \end{array}\right] \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_3 = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline 1 & 1 & \end{array}\right] \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_4 = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & \end{array}\right] \rightarrow \begin{bmatrix} 1, & 1 & 1 & 1 \\ 1 & 0, & 1 & 0 \\ 1 & 1 & 0, & 1 \\ 1 & 0 & 1 & 0, \end{bmatrix}$$

$$\mathbf{C}_5 = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & \end{array}\right] \rightarrow \begin{bmatrix} 1, & 1 & 1 & 1 & 1 \\ 1 & P, & 1 & 0 & 1 \\ 1 & 1 & P, & 1 & 1 \\ 1 & 0 & 1 & 0, & 1 \\ 1 & 1 & 1 & 1 & P, \end{bmatrix}$$

Figure 2.2 Double-cycling to produce 5 sub-matrices of ***C***
(a gnomon is added each time)

Notes on the figures

(i) The first matrix (which is given) is the 1×1 matrix $\mathbf{C_1}$ which has the single element 1.

(ii) The second matrix is formed from the first by adding the gnomon in the 'angle bracket' indicated by horizontal and vertical lines. That has been filled by first a 1-step cycling of the 1 from $\mathbf{C_1}$, to the right, and then a 1-step cycling of the 1 downwards. These two steps have respectively filled the R-arm and the L-arm of the gnomon, as shown in the middle column of Figure 2. The 'corner element', or 'next diagonal element', has been left vacant, awaiting further explanation.

(iii) Observe that the R-arm and the L-arm of the gnomon are vectorially equal.

(iv) In every case, the corner element of the gnomon is decreed to be 0. he reasons for this are discussed in **(vi).**

(v) When a prime fundamental cycle (see def. 2.2(i)) is reached, it is helpful to write P in the corner of the gnomon, where 0 should be, as shown in $\mathbf{C}_5$. Then the eye can run down the leading diagonal and easily spot where the primes are.

(vi) The reason for the 0 decree

If we were to cycle from $\mathbf{C_2}$ the 1s in row r_2 and column c_2, 1-step across and down respectively, and then cycle r_3 and c_3 for 1 step, as if from $\underline{\mathbf{1}}^{(1)}$, that would fill the corner with an element 1 in both cases. And if we were to continue cycling in this manner, cycling the single element of $\mathbf{C_1}$ repeatedly across and down, and the f.c.s $\overline{10}$ in the even rows and columns of $\mathbf{C_2}$, we would get the following pattern growing within **C** thus:

1 1 1 1 1 1 1 ...
1 0 1 0 1 0 1 ...
1 1 1 1 1 1 1 ...
1 0 1 0 1 0 1 ...
1 1 1 1 1 1 1 ...
… … …

Thus the matrix **C** would be filled with an endless pattern of $\mathbf{C_2}$ matrices, lining up, across and down, filling the columns and rows. No new (0,1)-patterns would be formed.

This is clearly not satisfactory, as we are trying to build a model of all the positive integers. That is why we decreed (in Definition 3.1) that, after $\mathbf{C_1}$, all leading diagonal elements shall be 0; that also makes it consistent with the cycling of *n*-words operated upon by the function kappa (see the Remark and Example below.)

We are taking pains to make this point, because we wish the reader to see that sometimes a 0 arises on the l.d. purely by row-cycling of (smaller) fundamental cycles, and sometimes a 1 arises which has to be rejected and replaced by a 0 (check this statement from Figure 2.1). The latter action has to be taken when and only when a new prime is being created. So our definitions and procedures actually create primes, as the sequence of matrices $\mathbf{C_n}$ is built by adding gnomons. That is a satisfying remark --- our decree only comes into play when a prime is created.

The l.d. plays a leading role (no pun intended) in our studies of primes using (0,1)-patterns. As already noted, we sometimes find it helpful to mark those rows whose cycle-number is prime, by writing P where they meet the l.d., rather than 0, as is done in $\mathbf{C_5}$**,** and remembering that this temporary switch has been made. And sometimes we mark l.d. elements with a following comma.

Study of the gnomons for matrices $\mathbf{C_3}$, $\mathbf{C_4}$, and $\mathbf{C_5}$ in the Figure will further elucidate all these remarks.

(iii) Now we can connect our $\mathbf{C_n}$ matrices with fundamental cycles and cycle-numbers, (which were defined above in Def. 2.1), thus:

Definitions 3.2:

(i) The fundamental cycle for row *n* is the bottom row of the completed matrix $\mathbf{C_n}$. It reaches to and includes the diagonal element of **C** in position *(n,n)*.

(ii) The cycle-number $\underline{\mathbf{n}}$ occupies row r_n and is obtained by cycling indefinitely the fundamental cycle in that row. We may use step-by-step (one or more elements at a time) cycling, or make whole cycle repetitions as the occasion demands.

Theorem 3.1:

(i) The matrix **C is** symmetrical about the l.d. That is, **C** is equal to its transpose.

(ii) The columns of **C** cycle downwards in the same way that the rows cycle across, with $r_n = c_n$, $\forall n$.

Proof: **(i)** follows immediately from the way that each row and column of **C** is constructed, by the double-cycling process. Every gnomon has equal arms.

(ii) follows from **(i).**

Remark: The definition and behaviour of the cycle-numbers in **C** is consistent with the original definition of the fundamental cycles. Their cycling, explained by means of enteger words and the kappa function, will clearly generalize too, as shown in the following example.

Example: The n-word (see row 3, Figure 1) for *n = 3* is $(3_1, 3_2, 3_3)$. Extending this word for a further 3 terms we get $(3_1, 3_2, 3_3, 3_4, 3_5, 3_6)$. Applying the kappa operator to this vector we get: (110,110) which is seen to be a cycling of the fundamental cycle (110) appended to itself; we may write this as $\mathbf{n^{(1)}_3} \ast \mathbf{n^{(1)}_3}$**,** and call $\ast$ the 'adjoin' operator.

The next three terms are $3_7, 3_8, 3_9$, which under kappa also map to 110 , constitue the second full cycling of the f.c. It is easy to show that this pattern must continue indefinitely, and so the cycling behavior of the enteger 3-word corresponds to row r_3 of matrix **C.**

As stated above, we can do the same for any n-word, and obtain under kappa a correspondence with row r_n of **C**.

4. FURTHER STUDIES OF ROWS IN C : POTENTIAL PRIMES AND STALACTITES IN C

4.1 Studies of the cycle-numbers in rows of C

Having shown how matrix **C** is derived, we are now in a position to study the (0,1)-patterns arising within it. We have already defined (see Def. 2.1) the pattern which signals a prime cycle-number **n**. Its fundamental cycle will begin with *(n-1)* 1s , and end with a 0 on the l.d. It will then cycle indefinitely to the right.

We now undertake a systematic study of the cycle-numbers **n** , through *n* = 1 to 7. It is necessary to observe so many examples, in order to bring out the salient properties of the **n**-sequences sufficiently deeply for the reader to understand the details of the proof of the twin primes conjecture given later.

We shall introduce the use of the Boolean 'product' operation ($\wedge$) to 'multiply' rows of **C,** and show how the products can be interpreted. Along the way we shall introduce several named concepts which play roles in the **TPC** proof to follow. In particular there will be the notions of *potential prime numbers* (symbol pP), *potential twin primes* (pTP or pT), *stalactites* (*j*-stal.) (in column c_j or **C**j), and *n-sieves*.

These concepts will be defined when we need them.

We shall need an example matrix to refer to, as we proceed from row to row, so we now present the 13×13 one (i.e. $\mathbf{C_{13}}$), which takes us down to row 13. Note that we have written the natural numbers along the top, and they can be used either as subscripts for the columns (c_j = col. j) or as the natural number corresponding to that column. Note also that we have placed commas after the l.d. elements, so that the fundamental cycles in the rows can easily be located. as with coordinates, we can write (n,j) to denote the position of the element in the n^{th} row (i.e. R_n or r_n) and the j^{th} column (i.e. C_j or c_j).

The Figure for $\mathbf{C_{13}}$ is as follows:

n\j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
1	1,	1	1	1	1	1	1	1	1	1	1	1	1
2	1	0,	1	0	1	0	1	0	1	0	1	0	1
3	1	1	0,	1	1	0	1	1	0	1	1	0	1
4	1	0	1	0,	1	0	1	0	1	0	1	0	1
5	1	1	1	1	0,	1	1	1	1	0	1	1	1
6	1	0	0	0	1	0,	1	0	0	0	1	0	1
7	1	1	1	1	1	1	0,	1	1	1	1	1	1
8	1	0	1	0	1	0	1	0,	1	0	1	0	1
9	1	1	0	1	1	0	1	1	0,	1	1	0	1
10	1	0	0	0	0	0	1	0	1	0,	1	0	1
11	1	1	1	1	1	1	1	1	1	1	0,	1	1
12	1	0	0	0	1	0	1	0	0	0	1	0,	1
13	1	1	1	1	1	1	1	1	1	1	1	1	0

Figure 4.1. The matrix $\mathbf{C_{13}}$

4.2 The cycle-number 1 (Study of row one)

We shall examine the cycle-numbers for $n = 1, 2, 3, \cdots$, and describe their properties with respect to various concepts, defining these as we go,. It is important to draw attention to many small details on the Figure. We begin with the cycle-number in r_1. Studies of r_2, r_3, etc. will appear later.

We have seen that this number has fundamental cycle $\underline{\mathbf{1}}^{(1)}$ = (1), which cycles indefinitely in row r_1 of **C** . Hence a **0** never occurs. In particular, **1** doesn't satisfy the definition of a prime cycle-number given in Def. 2.1: so **1** is not a prime cycle-number. This corresponds with the fact that thr natural number 1 is not prime.

Each **1** in r_1 indicates that $\kappa(1_j) = 1,\ for\ j = 1\ to\ \infty$.

4.3 Potential primes, pP

Let us introduce the pP concept now. We know that col. c_j will contain a prime cycle if and only if its column extends down to the l.d. with (j-1) 1s. Hence, as we construct the rows of **C** sequentially downwards, we shall know that c_j *is, or contains, potentially a prime, as long as its column does not acquire a **0** in it*. As long as this condition is met, we shall say that "c_j is pP", which is short for 'Col.j has, so far, only 1s from the top of **C**'. This will serve as our definition:

Definition 4.1: c_j ,(and the cycle-number it contains) is *potentially prime* (pP) as long as it does not acquire a 0.

Theorem 4.1: Every natural number greater than 1 is pP from row r_1.
Proof: Every column has first element 1; so, with the exception of 1 in c_1 (since **1** has already been shown to be non-prime), all cycle-numbers in columns c_j of row j>1 are pP s.
[N.B. We are simply saying that all of, 2,**3**,**4**,**5**,**6**,**7**, ... are potential primes.]

4.4 Stalactites in C

We now introduce the concept of a *'stalactite'* in the matrix **C** .

Definition 4.2: As we proceed from row to row down the matrix **C**, each time constructing a new cycle-number, the columns are sequentially filled with 0s or 1s.

(i) In every column the first element is 1, and a succession of 1s will occur in the column until a '*first zero*' 0 occurs. We shall call a growing sequence of 1s in a column an '*unstopped stalactite*' of the column. When it acquires its first zero, it will be called the '*final, or stopped stalactite'* of the column.
[N.B. We shall not always include the words *unstopped* or *final*, when it is clear from the context what is the current state of a stalactite. We shall sometimes call a stalactite in column *j* a *'j-stal'* .]

(ii) The stalactite whilst still '*growing 1s*'in column c_j has length equal to the number of 1s acquired so far. When it is stopped by the occurrence of a 0, its final length *l* is equal to the number of elements in the column down to and including the first zero.

The metaphor we have in mind, of course, is that matrix **C** is an underground cave, and the stalactites are growing down from the roof, as columns of 1s, until being 'stopped' by their first zeros.. We have found this notion to be a useful aid to understanding and writing about the occurrence of primes and twin primes, via the so-called stalactite algorithms.

Theorem 4.2:
(i) If a stalactite in column *j* (a *j*-stal.) reaches a length *(j-1),* at the next stage it will (a) reach the l.d. , (b) acquire a 0 (i.e. 'be stopped') and (c) signal that j is prime.
Further, in view of the symmetry of **C** w.r.t. the l.d., its transpose in the row r_j will be a prime fundamental cycle for the row,

and **j** is then a prime cycle-number.
(ii) We can say much better than (i). Recall that the first (j-1) elements of c_j are a palindrome (theorem. 2.1(i)). Hence if a *j*-stal. reaches a length $\lfloor j/2 \rfloor$ it will signal that *j* is a prime.

Proof: (i) follows directly from definitions given above.
(ii) follows from the palindrome law, theorems 2.1(i) and (ii).

We now continue with the analysis of cycle-numbers for $n = 1, 2, 3, \cdots$ and find out what happens to the stalactites as they grow.

4.3 The cycle-number 2 (Continuation of row studies)

Looking at r_2 in $\mathbf{C}_{13}$ (Fig. 4.1) we see that the fundamental cycle is (1,0), and it cycles along the row. Thus every alternate column c_j with even *j* acquires a 0 on its stalactite. All of these stalactites are 'stopped', their 'growth' ended, with final length 2.

In other words, columns 2,4,6,8, ... (this continues beyond $\mathbf{C}_{13}$ as far as we care to nominate) have stalactites of length 1 only. Only the first one, viz. that in c_2 , has reached the diagonal, where it signals that 2 is prime (and hence **2** is a prime cycle-number). All other j-stals of even j immediately lose their status of pP (potential prime) and are said to be non-prime, or nP. In short, mapping from cycle-numbers to numbers, all even numbers except 2 are non-primes (nPs).

The remaining stalactites (except in c_1), now have length 2, and are still pPs. They reside in columns with subscripts following the Arithmetic Progression A(3; 2), where 3 is the first element, and 2 the common difference, of the AP. Of course, we call these the odd integers greater than 1; but we wish to show how APs play a role in our analyses. We may now speak of the stalactites in the odd columns (thus far) as being pP of length 2.

Before we move to a study of **3** , we define the concept of 'n-sieve'.

4.4 The n-sieve

The repeating fundamental cycle in row r_n acts as a sieve being presented to the down-coming stalactites. If a stalactite arrives at this sieve, and finds a 0 immediately below it, that stalactite will be stopped. If the stoppage occurs on the leading diagonal (l.d.) then the pP becomes a P (i.e. a prime stalactite, prime column or prime cycle-number). If the stoppage occurs to the right of the l.d., the stalactite will no longer be a pP; and the corresponding cycle-number will be a non-prime, or an nP.

Whereas, if a down-coming stalactite arrives at the sieve, and finds a 1 immediately below it, it will acquire that 1, and in effect 'pass through' the sieve, increasing its length by 1. It becomes a longer pP and (to anthropomorphize the situation), it still has hopes of becoming a prime!

We shall call the sieve formed by all of **n** the *n-sieve*. It is the whole of row r_n. But we shall usually think of applying the sieve one f.c. at a time, cycling to the right, and observing what its head 0 is doing. It is helpful to imagine this right-wards motion of sieves.

We can **summarize** what we have found so far, in terms of our new language, thus:

(i) The first row of 1s in **C** indicates that all natural numbers greater than 1 are pPs, with the 1s forming unstopped stalactites in all columns.

(ii) The second row, of cycles $\overline{10}$ (i.e. cycle-number **2**) forms the 2-sieve. This sieve stops all the stalactites in the even subscripted columns; it allows all the unstopped stals. in the odd-subscripted columns to pass through, becoming pPs of length 3.

(iii) The 2-stal. (stalactite in c_2) is stopped with a 0 in the l.d., signalling that 2 is a prime. Hence **2** is a prime cycle-number, with stalactite of length 2.

(iv) The remaining pPs of length 2 are in cols. having subscripts in A(3,5), i.e. the arithmetic progression with first two elements 3,5.

Note: After a stalactite has been stopped, in subsequent sieving operations there is no need to insert indicated 0s and 1s below it. That is, if our only purpose is to discover whether a column is prime, we are content just to observe and record its stopped stalactite. Later we shall define a Boolean Product operation which enables us to derive a matrix from **C** which leaves element cells below stopped stalactites empty.

4.5 The cycle-number 3 (continuation of row studies)

3 occurs in row r_3 of **C**. Its fundamental cycle is (1,1,0), and **3** = $\overline{110}$.

We now apply the 3-sieve to the previous row, in order to determine which of the unstopped stalactites are stopped next, and which will pass through the sieve.

Observe that the only ones that can be stopped are those in columns with subscripts in A(3,6) = 3,6,9,12,15,18, ... where the 0s of $\overline{110}$ occur. But we already know from (ii) in the summary of 4.4 that all the even-numbered stalactites have been stopped in row r_2. So only odd-numbered stalactites, of length 2, present themselves to the 3-sieve. Hence the sieve will only stop those in columns c_j with $j \in 3,9,15\cdots =$ A(3,9) .

The first stoppage is in c_3 , when the stalactite there reaches the l.d., and therefore **3** is a prime cyclic-number.

And we know from (iv) of the summary that all unstopped-stals. (all pPs) lie in columns having subscripts in A(3,5).

Hence the stalactites of length 3, which have passed through the 2-sieve, are in columns with subscripts A(3,5)\A(3,9) = $A(5,11) \cup A(7,13)$, using set operations on A.P. sequences. The reason for the union of two APs is that $\overline{110}$ contains two 1s, each of which belongs to its own A.P. (one of the two shown) as the cycling takes place. Note that the A.P.s both have common difference 6. A useful notation for the above union is A(5,7**;** 6) showing in brackets just the initial elements of the progressions, with the shared common difference at the end.

It is also useful to combine the last two APs to form triples in the number line, to form a sequence of triples, which we shall call **S**. thus:
S = (5,6,7) , (11,12,13) , (17, 18, 19) , (23, 24, 25) , ... , *(n-1, n , n+1)* , ... , and note that the mid-term n is always a multiple of 6. We discussed triples of this nature in the Introduction.

A useful **notation** is the following:

In a triple of **S**, we call the number n-1 a *lower-6 (or l-6) number*. It is congruent to -1 mod 6. And the number n+1 is called an *upper-6 (or u-6) number*. It is congruent to +1 mod 6. Thus all primes greater than 2 are of one of these two types of number, either l-6 or u-6.

4.6 Potential twin primes (pTPs or pTs)

We note that all possible twin primes must occur in the triples of **S**, as *(n-1, n+1),* whenever both of $n \pm 1$ are primes. The first twin prime (TP) is (3,5) which has mid-term $n = 4$, and every subsequent TP has mid-term equal to $6m$, $m \geq 1$.

Thus every potential twin prime (a pT) after (3,5) is a pair of consecutive odd numbers of the form (l-6, u-6). This last remark is well-known in number theory. but we have proved it here, in the previous paragraphs, using our new language of stalactites. Summarising this in a lemma, we have:

> **Lemma 4.1:** All twin primes after (3,5) are pairs *(n-1, n+1)* occurring in the terms of **S** whenever both of $n \pm 1$ are primes. In row r_3 of **C** these pairs are all potential TPs, designated pTP or pT.

It will be evident that we can continue analyzing the cycle-numbers one at a time in sequence and determining when primes or twin primes occur. But that process will not, of course, prove that **P**s and **T**s must go on occurring indefinitely.

To illustrate the occurrences of stalactites in **C**, we show the $\mathbf{C_{13}}$ matrix again, with all the f.c.s shown 'below' the l.d., and stalactites shown up until their stoppages, above or on the leading diagonal . Note how the primes 2,3,5,7,11,13 have been formed, with their stoppage points labelled **P**.

n\ j	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1,	1	1	1	1	1	1	1	1	1	1	1	1
2	1	P,	1	0	1	0	1	0	1	0	1	0	1
3	1	1	P,		1		1		0		1		1
4	1	0	1	0,	1		1				1		1
5	1	1	1	1	P,		1				1		1
6	1	0	0	0	1	0,	1				1		1
7	1	1	1	1	1	1	P,				1		1
8	1	0	1	0	1	0	1	0,			1		1
9	1	1	0	1	1	0	1	1	0,		1		1
10	1	0	1	0	0	0	1	0	1	0,	1		1
11	1	1	1	1	1	1	1	1	1	1	P,		1
12	1	0	0	0	1	0	1	0	0	0	1	0,	1
13	1	1	1	1	1	1	1	1	1	1	1	1	P

Figure 4.2 The matrix $\mathbf{C_{13}}$ *with stalactites down to the l.d. stopped*

The reader should note patterns in the first 13 fundamental vectors, in the rows beneath the l.d. , especially noting the prime (0,1)-patterns (**P**s shown in the l.d. positions); and the fact that **2**, **4** and **8** have the same row-pattern, as have **3** and **9**. And note how all prime stalactites maintain their column runs of 1s down to the l.d. Observe also the pattern of columns (5,6,7) above the l.d.; it contains the first twin prime with mid-term a multiple of 6: the next is from columns (11,12,13).

5. THE OPERATIONS 'STAR ≡ ADJOIN' (*) AND 'CAP' ($\wedge$)

We now introduce two binary operations to be used to combine cycle-numbers in certain ways.

Definition 5.1: (The star, or adjoin, operation, (*))
Let two (0,1)-vectors be given, say **u** and **v**, having respective lengths *u* and *v*. Then **u***v** denotes *the adjoint vector,* which is formed by making the elements of **v** follow the elements of **u**, to form a new vector of length *u+v*.

Example: If **u** = (10), and **v** = (110), then **u*****v** = (10110) , with length 2+3=5.

Evidently, * is an associative operation, but is non-commutative.

Definition 5.2:
Let *n* be a scalar (an integer) and **u** be a (0,1)-vector.
Then *n**u** denotes the adjoint vector of *n* successive **u**s. Its length is *nu*.

Example: *n* = 3 and *u* = (10). Then *n*u* = (101010), of length 3x2 = 6.

Definition 5.3: (The 'cap', or 'product', operation, ($\wedge$))
Let $a,b \in \{0,1\}$. Then the product of *a* and *b* is denoted by $a \wedge b$ and called "*a* cap *b*", with the product table rules being:
0x0=0, 0x1=0, 1x0=0, and 1x1=1.

Below we shall call the operation the *Boolean Product, or BP,* and speak of *multiplying* two binary digits when computing it. Other names found for $a \wedge b$ in the literature of lattice algebras are 'the *meet* of a,b' and 'the *intersection* of a,b'. '*product*' is the most suitable term for our work with cycle-numbers; although the other names occasionally resonate nicely with what is happening.

In Section 6 we shall define the *BP* of two (0,1)-vectors, and also speak of *multiplying* two such vectors, which will then be called their 'joint vector'. It may also be called their *vector Boolean Product* or their *Coprime Product.*

6. JOINT VECTORS AND JOINT CYCLE-NUMBERS

6.1 Uses of the *BP* operation

We can use the *BP* binary operation of Definition 5.3 to combine two cycle-numbers, and then interpret the result in interesting and very useful ways.
As given in Definition 5.3, the rules for obtaining the *BP* of two binary digits are as follows: $0 \wedge 0 = 0,\ 0 \wedge 1 = 0,\ 1 \wedge 0,\ 1 \wedge 1 = 1$.

The cycle-numbers are infinite vectors, composed of 0 and 1 elements, and we next give explanations and rules for computing joint cycle-numbers.

6.2 Joint cycling of cycle-numbers

Two cycle-numbers, may be thought of as a pair of waves, moving along to the right in their rows of **C**. They may be cycling in, or out, of phase. We now define the process of finding a joint wave (or joint cycle-number) by combining them using Binary Product operations.

We define an operation on pairs of cycle-numbers (see definition 6.2) which determines, for any given pair (say **m** and **n**), their joint fundamental cycle and period.
Note that we allow the case $m = n$, when the two vectors cycle 'in phase' but with the joint vector having double their period. If the two vectors have $m \neq n$, then they cycle 'out of phase'; and their joint vector has period *mn*.
We then show how this operation can be usefully set to work on the rows of the matrix **C.** It is a well-defined product operation for combining two cycle-numbers in such a way that the result is itself a cycle-number.

Definition 6.2: (joint vectors, joint cycle-numbers)
The fundamental cycle (f.c.) of the joint cycle-number of **m** and **n** is defined to be: $[n*\mathbf{m}^{(1)}] \wedge [m*\mathbf{n}^{(1)}]$, where * is the adjoin operation, and $\wedge$ is the Boolean Product *BP* defined above. Note that the two vectors in square brackets are of equal length *mn*, so the Boolean operation is well-defined.

Evidently this definition can be generalized naturally to deal with joint cycling of more than two cycle-numbers. The *BP* is commutative and associative. The joint f.c. of one or more cycle-numbers cycles indefinitely in row *mn* of **C**, with its period being equal to *mn*.
When applied to two whole rows, it is meaningful to write $\mathbf{m} \wedge \mathbf{n}$ for the 'complete' joint cycle-number. It is itself a cycle-number (say **u**) which occurs in row $r_{mn} = r_u$.

Definition 6.3: (of 'measure')
Cycle-number **m** is said to *measure (*i.e. *it measures)* cycle-number **n** if and only if $\exists$ a positive integer v such that vm = n and $\mathbf{n}^{(1)} \wedge (v*\mathbf{m}^{(1)}) = \mathbf{n}^{(1)}$.
Lemma: If **m** *measures* **n**, with vm = n, then **v** *measures* **n**.

Example 1 (6.2): $(\mathbf{2} \wedge \mathbf{3})^{(1)}$ = [10,10,10] $\wedge$ [110,110] = (1 0 0 0 1 0) = $\mathbf{6}^{(1)}$
Note: If we allow ourselves to say that all cycle-numbers, as infinite rows in **C,** are of equal length (as they are in Cantor's 1-1 sense) we can write, for example, $\mathbf{2} \wedge \mathbf{3} = \mathbf{6}$.
In our work we are careful always to apply Definitions 6.2 and 6.3 to f.c.s, and then think of the result as cycling indefinitely.
Example 2 (
(i) **2** measures **6**, since 3x2 = 6 and **6** $\wedge$ (3*<u>**2**</u>) = **6** = (100010) $\wedge$ (10,10,10) .
(ii) **3** measures **6**, since 2x3 = 6 and **6** $\wedge$ (2*<u>**3**</u>) = **6** = (100010) $\wedge$ (110,110) .

Example 2: (an example of case *m=n=3*)
Let $\mathbf{m}^{(1)}$ **= (110) =** $\mathbf{n}^{(1)}$, Then their *joint vector* is (110110110) = $\mathbf{9}^{(1)}$**.**
This is the f.c. of **9** . So the f.c. is of length 9, and is the 'tonic' of the cycle-number **9** . The 'music of **9**' is also generated by the 'overtone' (110) = $\mathbf{3}^{(1)}$ **.**

Notes:

(i) We have not placed any restriction on gcd(m,n) in Definition 6.2. Both cases $m = n$ and $m \neq n$ are allowed. We wish to declare the period of the joint cycle to be mn, which it always is by Definition 6.2. If gcd(m,n) is greater than 1, there will be at least one shorter cycle vector which will also generate the whole of the joint vector. By analogy with musical notes (vibrating strings, say) we can call the fundamental cycle the *'tonic'* note of the joint vector, and any shorter generating cycle vector an '*overtone*'. Example 2 above gives an example of this case.

(ii) An equivalence relation: We shall say that two cycle-numbers are ρ -equivalent if they have the same infinite (0,1)-string in **C**. They need not have the same f.c.; for example, in $\underline{\mathbf{2}}\,\rho\,\underline{\mathbf{4}}$ the relatives have periods 2 and 4 respectively, being the lengths of their respective f.c.s. Note that ρ is an equivalence relation, which partitions the set of cycle-numbers into classes. The equivalence class containing $\underline{\mathbf{2}}$ is the set of all powers of $\underline{\mathbf{2}}$ under the Boolean Product.

(Elsewhere we have written $\underline{\mathbf{2}} \equiv \underline{\mathbf{4}}$, with the same meaning as $\underline{\mathbf{2}}\,\rho\,\underline{\mathbf{4}}$**.)**

6.3 Music of the integers

Much has been said in the mathematics literature about 'the music of the primes' (see [7], for example) particularly in connection with properties of the zeta function. We feel it is highly appropriate now for us to speak generally of '*the music of the integers*', in view of their cycling properties within the coprime matrix **C**. Every cycle-number vibrates in its row (with a 'characteristic sound', say). These sounds may be 'heard', emanating from every row and column, and also from every sloping diagonal line, on both upward and downward diagonals.

C may be partitioned into classes of rows each of which have the 'same sound'; for example, rows r_2, r_4, r_8, r_{16}, ... all have the same sound. Their (0,1) row-patterns to infinity are 1-to-1 identical (but their rhythms and emphases may be described differently)

The sound of any cycle-number clearly depends upon the composition of its fundamental cycle: but different pulses within their tonic and base cycles can be assigned or imagined.

Two or more cycle-numbers emit chordal sounds or polyphony, determined by the fundamental cycle of their joint vector. Indeed, there is so much cycling of patterns going on in matrix **C,** that one could imagine symphonies being composed, of selections of various infinitely repeating patterns.

6.4 Partial Boolean Row Products in the C matrix

Definition: We can derive a new matrix from **C**, which we shall call the Partial Boolean Product Sequence matrix, and denote it by **PBPS**(**C**). We define the nth row of this matrix to be the Boolean Product of the first n rows of **C**, applying Definition 6.2 each time, after row 1.

Thus the first 4 rows of **PBPS(C)** are: **<u>1</u> ;** **<u>1</u>**∧**<u>2</u> ;** **<u>1</u>** ∧ **<u>2</u>** ∧ **<u>3</u> ;** **<u>1</u>** ∧ **<u>2</u>** ∧ **<u>3</u>** ∧ **<u>4</u> :** and so on**.** Note that ∧ is an associative and commutative relation, so these *BP*s can be computed in any order of their terms. For example;

<u>1</u> ∧ **<u>2</u>** ∧ **<u>3</u>** ∧ **<u>4</u>** = **<u>1</u>** ∧ **<u>2</u>** ∧ **<u>4</u>** ∧ **<u>3</u>** , and since **<u>2</u>** ∧ **<u>4</u>** ≡ **<u>2</u>** we have the reduced result of the *BP* as **<u>1</u>** ∧ **<u>2</u>** ∧ **<u>3</u>** .

We can keep track of the pPs and pT's as we apply one *n*-sieve after another, by means of the Boolean ∧ operation which is used to 'join' the rows of **C** as we move downwards. A description of this sequential process now follows.

6.5 Computing the Partial Boolean Product Sequence; and the PBPS(C) matrix

We proceed as follows, applying the operations to rows of $\mathbf{C_{13}}$ as example.:

Row 1 of of the PBPS sequence is the same as row one of $\mathbf{C_{13}}$**.**

Compute the *BP* of row 1 and row 2 from $\mathbf{C_{13}}$, and place it in r_2 of the **PBPS** matrix. Next compute the *BP* of r_2 (i.e. r_2 of **PBPS**) and **<u>3</u>**, and place it in r_3 (of **PBPS**). Next compute the *BP* of r_3 and **<u>4</u>**, and place it in r_4 of **PBPS**. Continue likewise until (in our case) row 13 is reached. Clearly this sequence can be continued indefinitely from **C.**

We call this process '*Computing the Partial Boolean Product Sequence of* C_{13}' and the result is another (0,1)-matrix, which we shall call **PBPS**($\mathbf{C_{13}}$). Moreover, we can evidently compute **PBPS**($\mathbf{C_n}$) for any chosen value of n.

(In the general case we shall think of *n* tending to infinity, and operating on the cycle-joins, which are potentially infinite, to produce a doubly infinite matrix **PBPS**(**C**). We discuss this matter again below Figure 6.1, and in 6.4.)

First three rows of PBPS($\mathbf{C_{13}}$)

(1) row 1 of the **PBPS** matrix is the same as row 1 of the starting matrix.

(2) we compute row 2 of the **PBPS** matrix thus:

```
       row 1             1  1  1  1  1  1  1  1  1  1  1  1  ...
       row 2       ∧     1  0  1  0  1  0  1  0  1  0  1  0  ...
                         ------------------------------------
 joint  row, new r2      1  0  1  0  1  0, 1  0  1  0  1  0
growing stalactites in cols j  1     3     5     7     9     11    ... etc.
```

(3) we compute row 3 of the **PBPS** matrix thus:

```
   new row 2             1  0  1  0  1  0,  1  0  1  0  1  0 … etc.
    old row 3      ∧     1  1  0  1  1  0,  1  1  0  1  1  0 … etc.
                         ------------------------------------
 joint  row, new r3      1  0  0  0  1  0,  1  0  0  0  1  0 … etc.
pP stalactites in cols. j  1           5     7           11    ... etc
```

Figure 6.1 Computation of first three rows of ***PBPS($\mathbf{C_{13}}$)***

Observe that we have shown finally the first two cycles of $(2 \wedge 3)^{(1)}$, and from then on the (0,1)-patterns in the rows $\begin{pmatrix} r_2 \\ r_3 \end{pmatrix}$ repeat (i.e. cycle) with period 6. In particular note from (3) that the $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$-vectors (or g s) indicate the locations of potential primes (pPs), two in each cycle; and the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$-vectors (or f s) indicate where stoppages of stalactites in columns c_3 and c_9 respectively occur. The first is in the l.d., so it indicates a new prime cycle-number, namely **3** . The second indicates that the column c_9 ceased to be pP in row 3.

These notions are crucial for the development of our general algorithms.

6.6 Note on the processing of infinite sequences

We might be questioned about our assumption that we can 'multiply' two potentially infinite sequences, term-by-term, as we have done. Certainly, if we have two finite (0,1)-vectors of equal length, we can multiply them term-wise like this. To avoid thinking of 'whole infinities', we may observe that both our product terms are cycle-numbers, each cycling with its own fundamental cycle. They are cycling jointly in their two rows of **C**, and since their lengths are unequal, they are cycling out of phase. We only have to find their joint period, and their joint fundamental cycle, and apply the Boolean operation up to that point. That will tell us all we need to know, for thereafter all the numbers we require (indicating occurrences of stalactites, pPs and pT's in particular) will occur cyclically, with column subscripts in arithmetic progressions.

Taking **6.3** above for example, the first two fundamental cycles are $\underline{\mathbf{1}}^{(1)}$ = (1) and $\underline{\mathbf{2}}^{(1)}$ = (1,0).
It is immediate that their joint cycle period is 2, so we need only 'multiply' (i.e. take the BP of) (1,1) and (1,0), and we shall have the whole picture. We shall know that the first stalactite to pass through the 2-sieve is in c_1, and thereafter the stalactites of length 2 will be in columns c_j with $j = 3,5,7,...$, occurring 'endlessly' as first elements in the joint-cycles. Thus all odd numbered columns now contain pPs.

We are mindful of an assumption made by Bernhard Riemann (1825-1866) when he worried about the notion of 'endless' lines in space. He assumed 'unbounded but finite lines' to avoid the assumption of infinitudes. This led him to discover a consistent non-Euclidean geometry. Similarly, we can always think of 'unbounded but finite' cycle-numbers, and operate on chunks of them as joint cycles of finite period.

In the case of our (0,1)-pattern sequences, we avoid thinking of 'whole infinities' by considering only convenient 'chunks' of cycles, and milestones set along the pathway to infinity. Convenient and insightful milestones for our study are the so-called primorial numbers {2x3, 2x3x5, 2x3x5x7, 2x3x5x7x11**, ...** } with general term $p_k\#$ with the primorial (c.f. factorial) taken over the first k primes. Although the # -sign is in general use, we need a simpler symbol for numbers in this sequence, so we shall use $\mathbf{X_p}$ (capital *chi* ; mnemonic --- they are all multiple products), thus the primorial number sequence begins: $X_2 = 6$, $X_3 = 30$, $X_4 = 210$, $X_5 = 2310$..

We shall continue to use capital *chi* (i.e. **X**) when taking primorial products of our cycle-numbers, using Boolean multiplication, and if necessary write $\mathbf{X_n}$ for BPs.

7. RE-CASTING OF THE COMPUTATION OF PBPS(C) SEQUENTIALLY

We now continue our analysis of rows in **PBPS**(**C**), making use of what has just been said in the above notes, about processing potentially infinite rows of **C**.
We shall first recast and then extend the work of section 6, operating on the cycle-numbers directly. We apologize for the repetitive nature of some of this section, but we believe it is necessary in order to simplify the exposition and to introduce further concepts.

7.1 First three rows of PBPS(C) : re-casting of Figure 6.1

We now re-cast the procedures of Figure 6.1 in terms of the cycle-numbers and their fundamental cycles. Note that at each stage we only need to compute the fundamental cycle for the next row, beginning with $\mathbf{1}^{(1)}$ for row 1, and noting that it can be cycled as far as is required at each successive step.

We shall require Definition 6.2 for finding joint fundamental cycles and their periods. The two expressions in square brackets of the formula are labelled respectively A and B, and will appear in form 'A over B' in the applications below.

7.2 The complete re-cast of Figure 6.1 below shows how the fundamental cycles of rows 1,2,3 of **PBPS**(**C**) are obtained.

(1) **row 1** of the **PBPS**(**C**) matrix is the cycle-number **1** , using its f.c. $\mathbf{1}^{(1)}$**,** in **C**.

(2) **we compute** $\mathbf{r_2}$ of the **PBPS**(**C**) matrix from the f.c. of **1** ∧ **2** (see thm. 6.3(ii))

2 cycles of the f.c. of **1**	1 1	
1 cycles of the f.c. of **2**	1 0	(Boolean Product)
	----------	
new $\mathbf{r_2}$: joint f.c. $(\mathbf{1} \wedge \mathbf{2})^{(1)}$	1 0	(period 1x2 = 2)
unstopped stalactite col. *j*	1	

(3) **we compute** $\mathbf{r_3}$ of the **PBPS**(**C**) matrix thus:

3 cycles of the new r_2	1 0 1 0 1 0 = A	
2 cycles of the f.c. of **3**	1 1 0 1 1 0 = B	(take Boolean Products)
	-------------------	
new $\mathbf{r_3}$**:** joint f.c. $(\mathbf{2} \wedge \mathbf{3})^{(1)}$	1 0 0 0 1 0	(period 2x3=6)
unstopped stalactite cols. *j*	1 5	(potential primes in c_1 and c_5)
		(note the f in c_3 and the g in c_1 and c_5.)

Figure 7.1 Computation of first three rows (fundamental cycles) of ***PBPS(C)***

Notes about Figure 7.1 (There is some measure of repetition in these remarks)
(i) The expression (**1** ∧ **2**) ∧ **3** = $\mathbf{X}_2$.

Since Boolean 'multiplying' is associative, we can write the expression as (**1**∧**2**)∧**3**.
And it is evident from (2) that **1**∧**2** = **2** , so the expression found in (3) is
the f.c. of **2**∧**3** .= $\mathbf{X}_2$

(ii) The rows of PBPS(C) generally

Let us designate the n^{th} row of **PBPS**(**C**) by $\mathbf{S_n}$ Then our algorithm is computing the sequence $\{\mathbf{S_n}\}$ for n = 1,2,3,4, 5, ..., using in turn the fundamental cycles of $\underline{\mathbf{1}}$, $\underline{\mathbf{2}}$, $\underline{\mathbf{3}}$, $\underline{\mathbf{4}}$, $\underline{\mathbf{5}}$, from **C** to compute, in turn, the fundamental cycles of :

$\underline{\mathbf{1}}, (\underline{\mathbf{1}} \wedge \underline{\mathbf{2}}), (\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} \wedge \underline{\mathbf{3}}), (\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} \wedge \underline{\mathbf{3}} \wedge \underline{\mathbf{4}}), (\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} \wedge \underline{\mathbf{3}} \wedge \underline{\mathbf{4}} \wedge \underline{\mathbf{5}}), \cdots$

(iii) Reductions: Using the fact that Boolean 'multiplication' is associative and commutative, we note that, because of the equivalence relation for rows with identical (0,1)-strings, we only ever need to 'multiply' two rows both of which have prime *n*-values. Further, we only need to compute successive Boolean products of pairs of rows, and this we shall exploit in a later algorithm. [N.B. We work with fundamental cycles of rows, whilst actually, as explained above in 6.4, realizing that they can be cycled indefinitely.]

To illustrate, we show how to reduce $\underline{\mathbf{S_4}}$ **:**
thus:

$\underline{\mathbf{S_4}}^{(i)}$ is the f.c. of $(\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} \wedge \underline{\mathbf{3}} \wedge \underline{\mathbf{4}})$, which is equal to the f.c. of $(\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} \wedge \underline{\mathbf{4}}) \wedge \underline{\mathbf{3}} = (\underline{\mathbf{2}} \wedge \underline{\mathbf{3}})$ (using the facts that $\underline{\mathbf{1}} \wedge \underline{\mathbf{2}} = \underline{\mathbf{2}}$, and $\underline{\mathbf{2}}^{(1)}$ measures $\underline{\mathbf{4}}^{(1)}$, so $\underline{\mathbf{4}}^{(1)} = \underline{\mathbf{2}}^{(1)} * \underline{\mathbf{2}}^{(1)}$, where * means 'adjoin'. Cycling this is the same as cycling $\underline{\mathbf{2}}^{(1)}$, so $\underline{\mathbf{4}} \equiv \underline{\mathbf{2}}$. Finally, $\mathbf{S_4}^{(i)} = (\underline{\mathbf{2}} \wedge \underline{\mathbf{3}})^{(1)}$.

N.B. We don't write $\underline{\mathbf{4}} = \underline{\mathbf{2}}$, since they have different fundamental cycles. Using a musical anology, we could say that $\underline{\mathbf{4}}$ has 'tonic' of period 4, with an 'overtone' of period 2.

We now present as a lemma the algorithm incorporating these types of reduction.

Lemma 7.1: (reduced algorithm)

Let $\mathbf{S'_p}$ equal the row $\mathbf{S'_n}$ of **PBPS**(**C**) when *n* = *p* is prime.
The reduced algorithm for computing rows of the **PBPS**(**C**), is to compute (using f.c.s) the sequence $\{\mathbf{S'_p}\}$ = the rows *n* = 2, 3, 5, 7, 11, etc.

To fill in the rows for which *n* is non-prime, we note that if *p*,*q* are any two consecutive primes, the intervening nonprime rows $\mathbf{S'_{p+1}}, \mathbf{S'_{p+2}}, \cdots, \mathbf{S'_{q-1}}$ are all equal to $\mathbf{S'_p}$.

Stating the general step of the algorithm, we have:

(i) If p, q are consecutive primes, $\mathbf{S'_q} = \mathbf{S'_p} \wedge \mathbf{S_q}$, where $\mathbf{S_q} = \underline{\mathbf{q}}^{(1)}$ in **C**; (*)

(ii) All intervening rows are equal to the prime row immediately above them in **PBPS**(**C**), which is $\mathbf{S'_p}$. That is, $r_p = r_{p+1} = r_{p+2} = \ldots = r_{q-1}$. (**)

(iii) Repeat steps (i) and (ii), first replacing *p*,*q*, by *q*,*q'*, where *q'* is the next prime after *q*. And so on, ad infinitum.

Lemma 7.2:

Referring to line (**) in Lemma 7.1 above, we can conclude that if a stalactite in the matrix **PBPS**(**C**) has passed through the p-sieve, it will continue to pass through all the non-prime rows beneath it before meeting the q-sieve.

In order to clinch the proof of an infinity of primes, we shall prove later that there is always a pP stalactite of length $|\underline{\mathbf{p}}^{(1)}|$ in some column r with r>p.

Example: Computation of the f.c. of S'_5 in PBPS(C)

In *Figure 7.2* below, we show the computation of row 5, i.e. $\mathbf{S'_5}$ **,** using the formula of Lemma 7.1. Note that the joint cycle produced is of period (2x3)x5 = 30, and it involves the Boolean 'multiplication' of 5 cycles of $\underline{\mathbf{6}}^{(1)}$ (top row) to 6 cycles of $\underline{\mathbf{5}}^{(1)}$ (second row). Using Lemma 7.2 we have omitted cycle-number **4** from the calculation.

We compute the fundamental cycle for $\underline{\mathbf{30}}^{(1)} = [5*\underline{\mathbf{6}}^{(1)}] \wedge [\mathbf{6}*\underline{\mathbf{5}}^{(1)}]$, and display the results in the table below. Commas are inserted to show the out-of-phase cycling. Note that row r_{30} in **C** is equal to row r'_5 in PBPS(**C**).

5 cycles of $\underline{\mathbf{6}}^{(1)}$	1 0 0 0 1 0,1 0 0 0 1 0,1 0 0 0 1 0,1 0 0 0 1 0,1 0 0 0 1 0,
6 cycles of $\underline{\mathbf{5}}^{(1)}$	1 1 1 1 0,1 1 1 1 0,1 1 1 1 0,1 1 1 1 0,1 1 1 1 0,1 1 1 1 0, (5-sieve)
$\wedge \rightarrow$ f.c. of $\mathbf{S'_5}$	1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0,
pP columns	1 (5) 7 11 13 17 19 23 (25) 29

Figure 7.2 Computation of row 5 (fundamental cycle) of ***PBPS(C)***

This table tells us that by the time row r'_5 is reached, there are 8 stalactites still unstopped in the f.c. of the joint-cycle of $\underline{\mathbf{2}}$,$\underline{\mathbf{3}}$,$\underline{\mathbf{5}}$, in columns corresponding to 1, 7, 11, 13, 17, 19. 23, 29. The six 0s of the 5-sieve stopped only those stals. of length 4 growing in columns 5 and 25 (=5x5), each indicated by the *f*- pattern $\begin{pmatrix}1\\0\end{pmatrix}$ occurring in the two columns above them. The $\begin{pmatrix}0\\0\end{pmatrix}$ entries indicate the columns whose stalactites had been stopped before reaching row 4, and hence were not stopped in row 5. They occur in columns 2x5, 3x5, 4x5. 6x5, indicating that the stalactites in those columns were stopped respectively in rows 2,3,2,2 .

In c_5 the 5-stalactite reached the l.d., so **5** is a prime cycle-number. Note also that (5,7) can now be declared a T (since 7 must complete its journey to the l.d., by the palindrome law). And (11, 13) and (17, 19) remain as pT's. (23,25) loses its status of pT' by virtue of the stoppage in c_{25}. Note also the palindromic arrangement of the $\begin{pmatrix}1\\1\end{pmatrix}$ vectors in the interval [1, 29], in the row $\mathbf{S'_5}$ **.**

Lemma 7.3: A stalactite stoppage occurs in a sieved row iff an *f*-type 2-vector occurs.

Proof: A stalactite in **C** is a column of 1s from the top of the matrix. Repeated Boolean products of consecutive pairs of these leaves a column of 1 s in the PBPS(**C**) matrix, so when a sieve head 0 is met, an *f* 2-vector occurs. If a stoppage of that stalactite had occurred earlier in the column, there could only be a 0 in the position above the sieve head 0, hence no *f* 2-vector would occur.

Theorem 7.5: If an f 2-vector ends a stalactite on the leading diagonal, then it determines a prime (say p) in column c_p.Then the p-sieve in row r_p:

(i) does not stop any stalactite until the one in $c_{(p.p)}$;

(ii) thereafter it stops the stalactites in all columns $c_{p.q}$ where *q* is a power of p, or a prime greater than p.

(for example, if $p=5$ then the 5-sieve stops (after c_5) just the stalactites in columns 5x{5, 7, 11, 13, 17, 19, 23, 25, ,,, }).

Proof (i) and (ii): The head 0 of the p-sieve occurs in columns with subscripts pq, where q = 1,2,3,4,5,6,7,8,9,10,11,12,13, … When q is non-prime and greater than 1, the head 0 will be in a column with stalactite that was stopped earlier than in row p.

The following theorem is remarkable in that it shows that using only the information from *Figure 7.2*, extended to column 30, together with theorems already proved, we can decide on the ultimate state of all the pPs (and hence the pT's) remaining in the interval (5, 30), going all the way down in **C** to row 30. In short, we can 'read' the primes in $\mathbf{C_{30}}$ at this point, and hence obtain the f.c. of $\mathbf{S'}_{30}$. This fact is given as a theorem next. REFER TO FIGURE 7.3 BELOW FOR **PBPS($\mathbf{C_{30}}$)**.

Theorem 7.6:
The pPs indicated by unstopped stalactites in row 5 and columns 7,11,13,17,19, 23 and 29 all reach the l.d. in $\mathbf{C_{30}}$.
Hence these numbers indicate the rows of prime cycle-numbers in the interval (5,30) = (5, X_5).
Thus all of the pPs and pT's in the interval become Ps and T's respectively.

Proof: First we note that moving down to row r'_6 , the 6-sieve will not change any of the growing stalactites, since they all 'pass through it' because they earlier 'passed through' both the 2-sieve and the 3-sieve. Hence they become unstopped stalactites of length 6.
The next column to the right which contains a pP is c_7, so the stalactite in c_7 reaches the l.d., and is stopped there; so **<u>7</u>'** 'becomes' a prime cycle-number. And the next step is to apply the 7-sieve to the following pPs to the right of c_7 and in row 6.
By Theorem 7.5 above, it does not stop any of the remaining stalactites up to col. 30.

Readers are advised to follow all the steps in these arguments by referring to the 30x30 matrix $\mathbf{C_{30}}$ in Figure 7.3, and noting just where all the pPs are, and how and why they become stopped, or not, during the sieving process.

We also think it helps to understand the following device, which we now introduce and call *the triangle squeeze.* It also exemplifies Theorem 7.6 given above. Indeed, we could avoid all the following paragraphs, by just citing Theorem 7.6. However, we include them so the principles can be grasped through examples.

The triangle at this point of the sieving, is the one between points (7,7), (7,30) and (30,30) as shown in Figure 7.3. We find it helps to imagine running a left-hand finger down the leading diagonal from position (7,7), whilst simultaneously running a right-hand finger down column 30, from (7,30) to the position (30,30) . The left-finger is 'squeezing' the distance between the fingers as they move, reducing that distance finally to zero, when they both reach (30,30).
(Fig. 7.3 shows also the triangle from (19,19)).
As the left-finger moves down the l.d. from row-7-to-8-to-9-10 it encounters no finishing stalactite, because all of the stalactites in c_8, c_9, c_{10} were stopped before row 7 ; we know this from *Figure 7.2* by observing the 0s in these columns, in the row marked 'f.c. of **<u>S</u>'$_5$**'. Further, we know that the 7-sieve has no effect on any of the

stalactites in columns (8-30), since the sieve 0s after 7 occur in columns 14, 21, 28, all of whose stalactites were stopped before row 7 (because the column numbers are respectively 2x7, 3x7, 4x7, their stalactites being stopped respectively in rows 2,3,2). Hence all six of the remaining unstopped stalactites become 10-stals. by the time r_{10} is reached.

Moving the left-finger down one more step (putting on the squeeze) brings the c_{11} stal. down to the l.d., so 11 is prime. Now the 11-sieve cannot affect any of the stals. following column 11, because 2x11 is 22 and the stal. in that column was evidently stopped earlier, in row 2 (**2** is a measure of **22**). And cycling the 11-sieve once further takes us beyond the given interval, to 33, beyond the column 30 in the triangle squeeze.

The triangle squeeze is now, in effect, complete. We don't even need to move the left-finger further down the l.d. Because the remaining pP stals. *must* complete their run down to the l.d. The 11-sieve doesn't stop the c_{13} stalactite, which must then complete; then all unstopped remaining stals. (in cols. 17, 19, 23, 29) can only be stopped by their own sieves, where they become Ps. (Alternatively, we can deduce this by the Palindrome Law applied to each of these columns).

Thus all pPs become Ps, and all pT's become T's in the interval (5, 30). Q.E.D.

As noted, we could have shortened the above proof, by invoking the Palindrome Law when applicable.

We repeat that $\mathbf{\underline{S}'_5}$ has given us the f.c. of **30** and all the primes in (5, 30).

[**N.B.** Later all this must be understood generally, not in terms of specific primes.]

Before moving on to later cases, the next being $\mathbf{C_{210}}$, we present below the figure for **PBPS**($\mathbf{C_{30}}$), the partial Boolean row products matrix to r_{30}, and make comments upon it. Note in particular how the stalactites in the diagram show us where the primes in the interval (1, 30) are, and how they dominate the figure. Note also that after a stalactite is stopped by a 0 in its column, we do not print any subsequent 0s. This allows the prime stalactites to stand out starkly (one might say as 'prime ribs in the body of the natural numbers').

	0									1										2										3
S'$_n$\ n	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	P	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	1		P		1		1		0		1		1		0		1		1		0		1		1		0		1	
4	1				1		1				1		1				1		1				1		1				1	
5	1				P		1				1		1				1		1				1		0				1	
6	1						1				1		1				1		1				1						1	
7	1					(A)	P	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	(B)
8	1							0			1		1				1		1				1						1	0
9	1								0		1		1				1		1				1						1	0
10	1									0	1		1				1		1				1						1	0
11	1										P		1				1		1				1						1	0
12	1											0	1				1		1				1						1	0
13	1												P				1		1				1						1	0
14	1													0			1		1				1						1	0
15	1														0		1		1				1						1	0
16	1															0	1		1				1						1	0
17	1																P		1				1						1	0
18	1																	0	1		→		1			→			1	0
19	1																		P_A	0	0	0	1	0	0	0	0	0	1	0_B
20	1																			0			1						1	0
21	1																				0		1						1	0
22	1																					0	1					↓	1	0
23	1																						P	0	0	0	0	0	1	0
24	1																							0					1	0
25	1																								0			↓	1	0
26	1																									0			1	0
27	1																										0		1	0
28	1																											0	1	0
29	1																												P	0
30	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0	0	1	(C)

*Figure 7.3 The **PBPS(C_{30})** matrix, with 0s largely omitted.*
(Observe triangle (A)(B)(C), indicating the squeeze triangle from r_7)

Notes:

(i) The 5-sieve must, in its 1st cycle, stop the c'$_5$ stalactite, in view of the composition of the f.c. **<u>6</u>$^{(1)}$**. Similarly, in its 2nd cycle**,** it must by-pass the stalactite in c'$_7$. Thus c'$_5$ becomes a P and the next pP, in c'$_7$, passes through the 5-sieve and continues growing as a pP. Now it must continue growing until it reaches the l.d. and becomes a P: we say 'must' because there is no other prime between 5 and 7 (remember that c'$_7$ is the <u>next</u> pP after 5), so there is no prime-sieve formed that can stop the 7-stal before the l.d.

(ii) Note the form of the equilateral triangle marked P_A, P_B, P_C on figure 7.3, in relation to primes 19 , 23 and 29. This type of triangle has to occur in **PBPS**(**C**) as our algorithm moves the sieving process, from prime to prime, down the l.d. in a triangle squeeze.

Both these notes show how the algorithm, defined below, operates sequentially.

(iii)

All of the non-primes in (1, 30) have had their stalactites stopped by the time that the sieves 2- , 3- , and 5- have been applied (14 in row 2, 4 in row 3, and 1 in row 5). No subsequent sieve can do anything but stop its own corresponding stalactite.

(iv)

Note the triangular pattern formed between each P, its path down the l.d. to the next P, and the alternate path which goes right from the upper P until the next stalactite is reached, then down until the lower P is reached. (Trace these paths with a finger.) To demonstrate, we have filled in these paths between P=19 and P=23 and P = 29 with the 0s which should be there, and placed appropriate arrows in the diagram.

There is a sequence of these triangles formed as we move a finger; from P to P down the l.d. : and each triangle is equilateral, in the sense that each side takes the same number of steps to traverse.

(v)*

An important observation (call it comment *) is that there are no prime stalactites between c'_{19} and c'_{23}, and that the stalactite in column c'_{23} must proceed down to the l.d. since there is no sieve starting on the l.d. between the two Ps. Of course we know *now* that **23** is a prime cycle-number, but later, when presenting a general case, say for c'_k, we shall not know that *k* is prime, and shall refer the reader to this comment when claiming that c'_k's stalactite must grow on and down to the l.d.

(v)**

A second comment (say **) which we shall refer back to, is that in every case where a prime stalactite is formed, it occurs at the first $\begin{pmatrix}1\\P\end{pmatrix}$ [i.e. a $\begin{pmatrix}1\\0\end{pmatrix}$] in the P row, and the next 2x1 vector to occur is a $\begin{pmatrix}1\\1\end{pmatrix}$.in the P-row, indicating the next pP.

8. ON BLOCKS IN C, AND THEIR CYCLING PROPERTIES

All the elements of **C** are either 0 or 1. We can consider any given finite collection of them, having given positions, and call it a *general (0,1)-pattern*. Then we can ask: "Does this general pattern repeat somewhere in **C**?" Answers are given in the following theorems, with some particular repeating patterns being discussed first..

Theorem 8.1:

Let B(*r*,*s*) be the Block (a sub-matrix) consisting of the following set of elements in **C** : $\{c_{i,j} \mid i = 1, \dots ,r\ ;\ j = 1, \dots ,s\ \}$. We shall write B(r) in the case that $r = s$.

The following Blocks cycle (repeat consecutively) and indefinitely to the right in **C:**

(i) $B(2) = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$.

Proof: **1** and **2** cycle jointly with period 2, with B(2) cycling indefinitely.

(ii) $B(4,6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = B(p_3 - 1, X_2)$ (recall: $p_3=5$, $X_2 = 2.3 = 6$)

Proof: **2** and **3** cycle jointly with period 2x3 = 6.
And **4** ≡ **2** doesn't affect the joint cycle-pattern of B(3,6).

.

Note that if the Boolean produuct of the rows of B(4,6) be taken, the resulting vector is (1 0 0 0 1 0), which is equal to $\mathbf{6}^{(1)}$**,** the f.c. of the joint cycle of the two prime rows in the Block. The next theorem generalizes this example.

Theorem 8.2:
The Blocks $B(p_k, X_k)$ and $B(p_{k+1}-1, X_k)$, $k = 1,2,3, \ldots$ repeat consecutively and indefinitely to the right in **C**, with period X_k .

Proof: **(i)** The prime cycle-numbers in rows $p_1, p_2, \ldots, p_k$ cycle jointly with period X_k. The set of all cycle-numbers in rows 1,2,3, … , p_k cycles jointly with the same period (see section 6.3 discussion). Hence the set of columns (i.e. the whole block taken column-wise) which constitute the block $B(p_k, X_k)$, cycles indefinitely to the right in **C,** with period X_k..

(ii) The prime cycle-numbers in $p_1, p_2, \ldots, p_k$ cycle jointly with period X_k. Inclusion of any non-prime cycle-numbers, in other rows r_i with $i < p_{k+1}$, does not affect their joint cycle-pattern (the 'triangle squeeze' argument in Theorem 7.6 makes it clear why not).
Note that the Boolean product of all the rows in the Block is the f.c. of the joint cycle of all the prime cycle-numbers amongst the rows of the Block. □

Theorem 8.3: (Cycling of general patterns in C)
Let G be a given finite general (0,1)-pattern in **C**. Then:

(i) We can find an infinite sequence (not necessarily unique) of repetitions of G in **C**; and such that:

(ii) all elements of G cycle jointly in their rows, as a collection of Arithmetic Progressions having a common 'common difference'.

Proof: **(i)** Since G is finite, the pattern is bounded below (geometrically) by some row r of **C**; and it is bounded to the right by some column c of **C**. Hence it is contained in the block $B(r,c)$.
Since there is an infinity of primes, we can choose a value of k sufficiently large for the Block $B_k \equiv B(p_{k+1}-1, X_k)$ to contain the pattern G.
By Theorem 8.2, block B_k repeats consecutively and indefinitely to the right, carrying the pattern G within it. This proves (i). It is not necessarily a unique sequence for G, since, for example,
there is another such sequence, downwards in **C**, exchanging rows and columns in the argument and using the fact that **C** equals its transpose.
(ii) Since block B_k is cycling with period X_k, every element of it is cycling similarly. In particular, all elements of G cycle similarly, each in Arithmetic Progression, and all with the same common difference X_k.. □

Example: The pattern G $\equiv$ 1 1 1 1 occurs in **C_4**, within block B(4,6)

1 0 1

1 1

(see Theorem 8.1(ii) above). Hence pattern G occurs in each repetition of this block, and each element of the pattern occurs in an arithmetic progression with common difference $X_2 = 6$. For example, the only 0 in G occurs in successive positions in row r_2 as elements $c_{2,2}$, $c_{2,8}$, $c_{2,14}$, etc., the column numbers being in arithmetic progression.

9. A COUNTING ALGORITHM FOR JOINT CYCLES

9.1 The joint cycling of two vectors; their joint fundamental cycle and period

The following theorem gives the f.c. and period of two (0,1)-vectors which are cycling jointly. It is essentially a repeat of Theorem 6.3, in a form needed for the counting algorithm given below.

Theorem 9.1: ***(Joint cycling of two vectors ... see also Thm 6.3)***
Let $\underline{\boldsymbol{u}}$ be a (0,1)-vector of length u, and $\underline{\mathbf{v}}$ be a (0,1)-vector of length v.
Let $\underline{A}$ = u*$\underline{\boldsymbol{v}}$ be the vector $\underline{\boldsymbol{v}}$*$\underline{\boldsymbol{v}}$*$\underline{\boldsymbol{v}}$* ...* $\underline{\boldsymbol{v}}$*$\underline{\boldsymbol{v}}$, i.e. u adjoined, or cycled, $\underline{\boldsymbol{v}}$ s,
and $\underline{B}$ = v*$\underline{\boldsymbol{u}}$ be the vector $\underline{\boldsymbol{u}}$*$\underline{\boldsymbol{u}}$*$\underline{\boldsymbol{u}}$* ...* $\underline{\boldsymbol{u}}$*$\underline{\boldsymbol{u}}$, i.e. v adjoined, or cycled, $\underline{\boldsymbol{u}}$ s.
Then the lengths of $\underline{A}$ and $\underline{B}$ are both equal to $uv = w$, say. And the joint vector $\underline{A} \wedge \underline{B}$ cycles with period w.
(proof is evident from the construction of $\underline{A}, \underline{B}$: see Thm. 6.3)

9.2 Counting types of element-pairs in the f.c. of two jointly cycling vectors

When two (0,1)-vectors cycle jointly, their elements pair-up in four different ways, from $\underline{\text{A}}$ and $\underline{\text{B}}$, to be Boolean multiplied, viz. (0,0), (0,1), (1,0) and (1,1).

The next theorem shows how the frequencies of occurrence of these four pair-types can be counted within $\underline{\text{A}} \wedge \underline{\text{B}}$ when producing the fundamental cycle of the joint vector.

Theorem 9.2: ***(Algorithm for counting pair-types)***

Consider the sequence of ordered pairs from $\underline{A}, \underline{B}$: $\{(a_1, b_1), \cdots, (a_w, b_w)\}$,
where $\underline{A}$ and $\underline{B}$ are the vectors defined in Theorem 9.1.
.
All the paired elements are 0 or 1, so the pairs can be classified as follows:

$$d = (0,0),\ e = (0,1),\ f = (1,0),\ g = (1,1).$$

N.B. Later we use the transposed forms, i.e. the corresponding 2x1 vectors, or 2-vecs. as we shall call them.

The following is an algorithm for counting the numbers of pairs in each class, within the fundamental joint-cycle..

Let u_0, u_1 be respectively the numbers of 0s and 1s in the vector $\underline{\text{u}}$: and
$v_{0,}\ v_1$ be respectively the numbers of 0s and 1s in the vector $\underline{\text{v}}$.

Assign the generating function expression $(u_0 + u_1 i)$ to $\underline{A}$, and
_____assign the generating function expression $(v_0 + v_1 j)$ to $\underline{B}$.

Then the algebraic product below generates the numbers we require:

$$(u_0 + u_1 i)(v_0 + v_1 j) = u_0 v_0 + u_0 v_1 j + u_1 v_0 i + u_1 v_1 ij \quad (*)$$

The coefficients on the right-hand side of (*) supply the class totals we require.
Thus: $\#d = u_0 v_0, \quad \#e = u_0 v_1, \quad \#f = u_1 v_0, \quad \#g = u_1 v_1$.

(i) Proof of a special case of Theorem 9.2:

We begin the proof by proving a lemma for a special case of the theorem.
Then the lemma below is applied successively to prove the theorem generally.

Lemma 9.2: Let ***u*** have one 1 and (*u*-1) 0s ;
and ***v*** have one 1 and (*v*-1) 0s ;
then in the fundamental cycle of the joint cycle
of **u** and **v** there is one 1 and *uv*-1 0s.

Proof:

The joint cycle has tonic (or fundamental) period $w = uv$,
and $\underline{A}$ and $\underline{B}$ are:

$$\underline{A} = \underline{u} * \underline{u} * \cdots * \underline{u} \quad (v \text{ adjoins of } \underline{u})$$

$$\underline{B} = \underline{v} * \underline{v} * \cdots * \underline{v} \quad (u \text{ adjoins of } \underline{v}) .$$

In the string of 2-vecs in $\underline{A} \wedge \underline{B}$, every element of ***u*** occurs once and once only, paired with every element of ***v***. Hence the 1 in ***u*** pairs up just once with the 1 in ***v***, in one of the *uv (=w)* possible pairs. When the BP $\underline{A} \wedge \underline{B}$ is computed, that pair yields $1 \wedge 1 = 1$. All other of the *w-1* pairs have one or two 0s, so their BPs are 0.

Note that the generating function formula would have given this result, from the product coefficient $u_1 v_1$. Therefore we have proved the Lemma, and confirmed the theorem for this special case. □

(ii) Proof of the general theorem 9.2:

There are u_1 1s in vector ***u*** , and v_1 1s in vector ***v***. If we select any 1 from ***u*** and any 1 from ***v***, we can apply the above Lemma and find that just one concurrence of the two 1s will be found in the product set $\underline{A} \times \underline{B}$. If we then select the same 1 from ***u***, and another (different from the first) 1 from ***v***, again we shall find, by the Lemma, another concurrence (1,1) in the joint cycling of ***u*** and ***v***. Continuing thus until all 1 s in ***v*** have been considered, we shall find v_1 concurrences (1,1).

Now we may choose another (different) 1 from ***u***, and repeat the above procedures to find v_1 further concurrences. Repeating this again and again, we shall find a total of $u_1 v_1$ concurrences; which confirms the total *#g* (coeff. of *ij* in the g.f.) as claimed in the given generating function formula, in (*).

Continuing with the main theorem, we may proceed exactly as above but with the 0s from the two vectors ***u*** and ***v***. This leads to confirmation of the formula for $\#d = u_0 v_0$ (numerical term in the generating function) for the number in the class of (0,0) concurrences.

Repeating this procedure again, choosing 0s from $\underline{u}$ and 1s from $\underline{v}$, we confirm the formula $\#e = u_0 v_1$ (coeff. of *j* in the g.f.) for the number in the class of (0,1) concurrences.

Finally, choosing 1s from **u** and 0s from **v**, we confirm that $\# f = u_1 v_0$ (coeff. of *i* in the g.f.) for the number in the class of (1,0) concurrences. □

A simple, useful corollary to the theorem follows immediately:

Corollary of Theorem 9.2:
The class numbers are invariant to any permutations of the elements in the vectors **u** and **v**.

Proof of Corollary: Permuting the elements in **u** , and in **v** , does not change the combined generating function for #d, #e, #f, #g . □

Example:
We give one short example to illustrate the somewhat complicated procedures explained above; it also shows a simple way of laying out the algorithm in action.
N.B. We have displayed the (0,1)-correspondences in 2x1 columns.

Vectors	**Period of joint cycle**	**Generating functions**	**Coefficient products**
$\underline{u} = (1,0)$	$w = uv = 6$	$1 + 1\,i$	| \\ / |
$\underline{v} = (1,1,0)$		$1 + 2\,j$	| \\ / |

------------- ------------------ **Type frequency table**

$v * \underline{u} = \underline{A} = (1,0,1,0,1,0)$

$u * \underline{v} = \underline{B} = (1,1,0,1,1,0)$

$$\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

		d	*e*	*f*	*g*
	type:	*d*	*e*	*f*	*g*
Joint g.f.: 1+2*j*+1*i*+2*ij* :	freq.(#)	1	2	1	2

These frequencies may be checked directly from the given vectors $\underline{A}$, $\underline{B}$.

9.3 General formulae for the successive joint-cycles computed in PBPS(C)

The **PBPS(C)** matrix is obtained by successively computing the joint-cycles of rows $\underline{\mathbf{p}}_{\mathbf{k-1}}$ and $\underline{\mathbf{p}}_{\mathbf{k}}$ in **C,** where p_i is the *i*th prime. The sequence of these joint-cycles occurs in rows r_2 , r_6 , r_{30} , $\cdots$, r_m where $m = X_k$, the k^{th} primorial . See Section 7 for details and examples.

We shall now apply the methods of the algorithm of theorem 9.2 to find formulae for #d, #e, #f, #g in the Primorial Blocks. For convenience we write these frequencies respectively as d_k, e_k, f_k, g_k . And we note that the rows from which the 2-vecs are counted are the last row-pairs in the block $B(p_k, X_k)$.

Theorem 9.3: (c.f. Section 11) The general formulae for #d, #e, #f and #g from the last two rows, $\mathbf{S'_{k-1}}$ and $\mathbf{S'_k}$ **,** in the block $B(p_k, X_k)$ in matrix **C** are::

__**General formulae**__: #d $\equiv d_k = (X_{k-1} - X_{k-1}^{(-1)})$, where $X_k = \prod_{n=1}^{k} p_n$,

and $X_k^{(-1)} = \prod_{n=1}^{k} (p_n - 1)$;

#e $\equiv e_k = (X_k - X_k^{(-1)}) - X_{k-1}$;

#f $\equiv f_k = g_{k-1} = X_{k-1}^{(-1)}$;

#g $\equiv g_k = X_k^{(-1)}$.

Proof: This follows directly from the method of generating these frequencies, as follows. Using induction, we check that the theorem is true for *k=2*. Then we assume the above formulae to be true from *k=2* to *k=k-1* .
Using the methods of theorem 9.2, we have to find first the generating function for row r_{k-1} in B_k, which by the assumption, is $u_0 + u_1 i$, with $u_0 = X_{k-1} - X_{k-1}^{(-1)}$ and $u_1 = X_{k-1}^{(-1)}$. Then we find the generating function for r_k , which is $v_0 + v_1 j$, $v_0 = 1$ and $v_1 = p_k - 1$.
Applying Theorem 9.2, we find that $d_k = u_0 \,.\, v_0 = (X_{k-1} - X_{k-1}^{(-1)})$,

$$e_k = u_0 \,.\, v_1 = (X_k - X_k^{(-1)}) - X_{k-1},$$
$$f_k = u_1 \,.\, v_0 = X_{k-1}^{(-1)},$$
$$g_k = u_1 \,.\, v_1 = X_k^{(-1)} \quad .$$

Each is in accord with its formula as given above. Hence by induction, the formulae are true for all values of k > 1. □

In the following figure (Fig. 9.1) we tabulate the frequencies of *d,e,f,g* from blocks B_k for *k = 1 to 5,* using the general formulae to compute them.

The generating functions are given in columns 3,4 In the final four columns, we list the corresponding frequencies of *d,e,f,g* which arise in the sequential sieving of blocks B_k. Recall that *d,e,f,g,* are respectively:
$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ occurring in rows $\begin{pmatrix}r_{k-1}\\r_k\end{pmatrix}$ of **PBPS**(**C**): hence they are calculated by the sequence of computations of $\mathbf{S'_k} = \mathbf{S'_{k-1}} \wedge \underline{p}_k$.

k	p_k	gen. functions		#d	#e	#f	#g	#g/#f
1	2	$0 + 1i$	$1 + 1j$	0	0	(≡ 1)	1	1
2	3	$1 + 1i$	$1 + 2j$	1	2	1	2	2
3	5	$4 + 2i$	$1 + 4j$	4	16	2	8	4
4	7	$22 + 8i$	$1 + 6j$	22	132	8	48	6
5	11	$162 + 48i$	$1 + 10j$	162	1620	48	480	10

Formula for the last column: $\dfrac{\#g}{\#f} = (p_k - 1)$.

*Figure 9.1 Table for d,e,f,g frequencies in prime row f.c .s of PBPS(**C**)*

Note that the number #g of pPs passing through the *k*-sieve is always *(p_k-1)* times the number *#f* of stalactite stoppages. This ratio increases rapidly with B_k.

10. PROOF THAT THERE IS AN INFINITY OF PRIMES (THE PC)

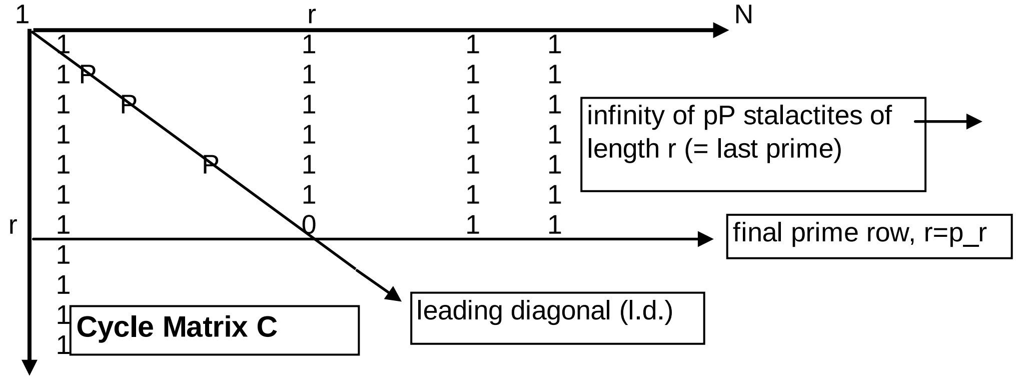
It has been known that there is an infinity of prime numbers since Euclid proved it as Proposition IX. 20 of his *Elements of Geometry* (c. 300 B.C.).

We wish, however, to prove the proposition using the tools that we have developed from the cycle-number matrix **C**.

Theorem 10.1: The list of primes in the sequence of natural numbers is infinite.

Proof: We suppose first that the theorem is not true, so there is in fact a last prime, say p_r. Then we deduce a contradiction.

Construction: The following diagram shows an upper-left portion of the cycle-number matrix **C,** with the positions of some of the primes indicated by Ps on the leading diagonal. The primes are actually in rows R2, R3, R5, … , R*r*, where $r{=}p_r$ is the last prime.



Let X_r be the primorial $p_r\, p_{r-1} \ldots p_3\; p_2\, p_1 = p_r\#$.

Define a Block B(u,v) to be the submatrix of **C** which starts in the top left corner of **C** and has u rows and v columns**.**

The block $B(p_r,X_r)$**,** in the cycle-number matrix **C**, cycles unchanged and indefinitely to the right, and it contains an infinity of stalactites of unstopped length $r = p_r$. In particular (as proof), all first columns of the cycled Block contain copies of the first r elements from column C1 of **C,** which are all 1s**:** also we note that each block-cycle contains as many such unstopped stalactites (length r) as there are in the first block-cycle, but we do not need to assume more than the one in column 1 of the second block-cycle.

The first of these stalactites can only be stopped by the 0 in the head of some prime-sieve in a row below p_r (i.e. below Rr). (They have already passed all the prime-sieves in rows Rn for n = 1,2,...,r.) But our starting assumption denies the existence of such a sieve below Rr.

Hence all the unstopped stalactites to the right of column C*r* will continue growing downwards, until the left-most one inevitably reaches the l.d., thereby indicating a new prime in its column.

This new prime is greater than p_r (lower than Rr in **C**), which is contrary to our supposition. So our theorem is proved, by *reductio ad absurdum*.

The list of prime numbers never ends. □

Comments: It will have occurred to the reader that this proof has mirrored some of the ideas used in Euclid's two thousand five hundred year-old proof. Indeed, it would have been strange if it hadn't, because we both deal with basic properties of the primes. But our definitions of the natural numbers and the primes differ markedly from his, and the language of the proof is quite different.

In particular we do, in effect, look inside a cycle-number (i.e. study its internal (0,1)-patterns), and we use notions such as pPs (potential primes, unstopped stalactites), prime-sieves, and Block-cycles, to present our arguments.

Comment: (A simple alternative proof of the PC?)
If the primes were to end at **p**, the matrix **C** could not go on being row-equivalent, through the growing set of cycle-numbers, to the set of natural numbers **N**. Indeed, no new cycle-numbers (where 'new' means having a (0,1) row-pattern different from all the rows above it in **C**) could be produced after **p** .

Thus the creation of the sequence of $\mathbf{C_n}$ matrices, adding gnomons with 0 corner-elements (decreed) forces the list of prime cycle-numbers, and hence primes and non-primes, to grow indefinitely long, maintaining the exact correspondence with the list of natural numbers according to their values and 'coprimeness' relations.

Comment and lemma on the radicals r(n) of cycle-numbers n:

When computing the rows of the **PBPS(C)** matrix above, in Sections 6 and 7, we have often taken advantage of the notion of equivalent pairs of rows, meaning that they both have the same (0,1) row-patterns. We shall close this Part I by giving further examples, and by relating the type of equivalence to the notion of *'having equal radicals'*. (see [8, p.6] for definition of *radical.*)

Examples: (i) **2**, **4**, **8**, **16**, ... have the same cycle-number pattern as **2** .
3, **9**, **27**, **81**, . have the same cycle-number pattern as **3**
10, **100**, **1000**, have the same cycle number pattern as **10** .

(ii) Consider the example for **10**, just given. Now 10 = 2x5. That is, the radical r(10) = 10. Likewise, for cycle-numbers, We can say that, since **10** = **2** ∧ **5** , the radical of **10** (under ∧) is **10** .
Hence all of **10**, **20**, **40**, **50**, **80**, **100**, **200**, and so on,
i.e. all numbers composed of both **2** and **5** , have the same radical and hence the same cycle-number pattern as **10**

(iii) **6** = **2**∧**3** ; hence all of **6**, **12**, **18**, **24**, etc. have the same radical and cycle-number pattern as **6** .
Lemma: All numbers of form $2^i 3^j$, $\forall\, i > 0,\ j > 0$ have the same radical; and their corresponding cycle-numbers all have the same (0,1) cycle- number pattern as **6** . (Clearly this generalises.)

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PART II - THE MATRIX C′ AND PROOF OF THE TWIN-PRIMES CONJECTURE

11. DEFINITION AND PROPERTIES OF C′

11.1 Derivation of C′ from C

In Part I we defined the matrix **C,** and after defining the Boolean Product on its elements, we were able to produce a matrix **PBPS(C)** ('Partial Boolean Product

Sequence') which then enabled us to study and count occurrences of prime numbers within the so-called primorial intervals $I_p \equiv (p, \mathbf{X_p})$, where $\mathbf{X_p}$ is the primorial function on p. We now generalize these procedures, in a sense, by deriving a matrix called **C′** from **C**, which will enable us to study and count occurrences of twin primes in similar ways. Thus we shall be led to our proof of the twin primes conjecture.

We begin with a definition of elements of **C′**, which are denoted by $c'_{i,j}$ in terms of those of **C**, denoted by $c_{i.j}$.

Definition 11.1: The *i,j*th element of **C′** is $c'_{i,j} = (c_{i,j} \wedge c_{i,j+2})$ for all $i > 0$ and $j > 0$. Evidently, since all elements of **C** are 0 or 1, then so are all elements of **C′**.

Lemma 11.1: Denoting rows and columns in **T′** by r'_i and c'_j respectively,

(i) $c'_j = c_{i,j} \wedge c_{i,j+2}$;

(ii) $c'_{1,1} = 1$; $c'_{i,i} = 0$ for $i>1$; so l.d. of **C′** equals l.d. of **C** ;

(iii) **C′** is not symmetric w.r.t. its l.d**.**

Proof: (i) and (ii) are immediately evident.

(iii) $c'_{4,3} = 1$ whilst $c'_{3,4} = 0$; this is a counter-example to symmetry.

11.2 The first five rows of C′

As we did with **C,** we shall show how the rows of **C′** can be computed sequentially. We do this with reference to the first five rows in the following Figure.
We define the *n*th row of **C′** to be $c'_{\mathbf{n}}$ and call it the *n*th c'-cycle-number.

Figure 11.1: *The first five fundamental cycles in **C′**, derived from n-words*

<u>***n***</u>	<u>***n***</u>$^{(+2)}$**-word**	κ **(**<u>**n**</u>$^{(+2)}$**)**	**Fundamental cycle** $c'^{(1)}_n$
1	**(1_1, 1_2, 1_3)**	**(1 1 1)**	**(1)**
2	**(2_1, 2_2, 2_3, 2_4)**	**(1 0 1 0)**	**(1,0)**
3	**(3_1, 3_2, 3_3, 3_4, 3_5)**	**(1 1 0 1 1)**	**(0,1,0)**
4	**(4_1, 4_2, 4_3, 4_4, 4_5, 4_6)**	**(1 0 1 0 1 0)**	**(1,0,1,0)**
5	**(5_1, 5_2, 5_3, 5_4, 5_5, 5_6, 5_7)**	**(1 1 1 1 0 1 1)**	**(1,1,0,1,0)**

Figure 11.1 shows the natural numbers 1 to 5 in column 1, then a column of so-called <u>n</u>$^{(+2)}$-words followed by a column of these operated upon by κ, and finally a column of the promised first five fundamental cycles. Note that an <u>**n**</u>$^{(+2)}$**-word** is obtained from its corresponding <u>**n**</u>**-word** by extending the sequences of entegers by two elements, in the manner shown. Then κ is applied, and finally, using definition 11.1, the *n* elements of $c'^{(1)}_n$ are computed by Boolean multiplication (BP).

Lemma 11.2: The vector $c'^{(1)}_n$ cycles with period *n* .

Proof: The jth element of the vector is $c'_{n,j} = c_{n,} \wedge c_{n,j+2}$

$= c_{n,j+n} \wedge c_{n,j+2+n}$

$= c'_{n,j+n}$, for all *n*.

Hence proof.

Evidently, we can continue these computations indefinitely downwards, and then cycle each fundamental cycle indefinitely to the right, and this computes the matrix **C′**, which we have earlier called the twin primes matrix.

Clearly the set of rows in **C′** can be set one-to-one with the natural numbers, so the set of their fundamental cycles provides another model of the natural numbers.

Lemma 11.3: The fundamental cycle in row r'_n of **C′**, when $n = p$ is prime, has elements 1,1,1, ... ,1,1 (x(p-3)) followed by 0,1,0. It cycles with period n.

Proof: This follows immediately from the construction method given in Figure 11.1.

In Lemma 11.1 we showed that **C′** is not symmetric, so we might ask whether or not it is doubly cyclic, as is **C**. We have shown already that the nth row cycles with period n. The following lemma proves that the columns cycle too, but with irregular periods, hence C' is not symmetric.

Lemma 11.4: The nth column of **C′** is cyclic, with period $n(n+2)$. **C′** is doubly cyclic.

Proof: Consider the nth column in **C′,** namely c'_n . It is equal to the Boolean product of columns c_n and c_{n+2} in **C**. It follows immediately that the c'_n cycles as the joint cycle of those two columns (see Theorem 9.1; cycling of pairs of columns in **C** is the same as cycling their corresponding rows). Thus c'_n cycles with period $n(n+2)$, and with f.c. as computed from the two columns in **C** by the methods of Section 9.

We now show, with minimal comment, two diagrams (Figures 11.2, 11.3) constructed to show some properties of **C′** diagrammatically. They may be compared directly with the equivalent ones constructed in **Part I** from **C,** *viz.*.Figures 4.1 and 4.2.

n\j	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
n													
1	**1,**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
2	**1**	**0,**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
3	**0**	**1**	**0,**	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**1**	**0**	**0**
4	**1**	**0**	**1**	**0,**	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0**	**1**
5	**1**	**1**	**0**	**1**	**0,**	**1**	**1**	**0**	**1**	**0**	**1**	**1**	**0**
6	**0**	**0**	**0**	**0**	**1**	**0,**	**0**	**0**	**0**	**0**	**1**	**0**	**0**
7	**1**	**1**	**1**	**1**	**0**	**1**	**0,**	**1**	**1**	**1**	**1**	**0**	**1**
8	**1**	**0**	**1**	**0**	**1**	**0**	**1**	**0,**	**1**	**0**	**1**	**0**	**1**
9	**0**	**1**	**0**	**0**	**1**	**0**	**0**	**1**	**0,**	**0**	**1**	**0**	**0**
10	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0**	**1**	**0,**	**1**	**0**	**0**
11	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0,**	**1**	**1**
12	**0**	**0**	**0**	**0**	**1**	**0**	**0**	**0**	**0**	**0**	**1**	**0,**	**1**
13	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**0**	**1**	**0**

Figure 11.2 The matrix $\mathbf{C'_{13}}$

n'\j	1	2	3	4	5	6	7	8	9	10	11	12	13
1'	1,	1	1	1	1	1	1	1	1	1	1	1	1
2'	1	0,	1	0	1	0	1	0	1	0	1	0	1
3'	0	1	0,		1		0		0		1		0
4'	1	0	1	0,	1						1		
5'	1	1	0	1	0,						1		
6'	0	0	0	0	1	0,					1		
7'	1	1	1	1	0	1	0,				1		
8'	1	0	1	0	1	0	1	0,			1		
9'	0	1	0	0	1	0	0	1	0,		1		
10'	1	0	0	0	0	0	1	0	1	0,	1		
11'	1	1	1	1	1	1	1	1	0	1	0,		
12'	0	0	0	0	1	0	0	0	0	0	1	0,	
13'	1	1	1	1	1	1	1	1	1	1	0	1	0

Figure 11.3 The matrix $\mathbf{C'_{13}}$ *, showing stalactites in the upper triangle*

Note from Figure 11.2 how the columns are constructed, as shown in the proof of Lemma 11.4 above. In particular, note the lengths of the stalactites in columns Cj' for j = 1 to 13. Two examples are C7′ = C7 ∧ C9 = **63** which is measured by **3** and so has stalactite 110, of length 3; and the second example is C5′ = C5 ∧ C7 = **35** which is measured by **5** and so has stalactite 11110, of length 5 and down to the l.d.

A few moments thought shows that only those columns Cj' in the **C′** matrix which result from the BP of columns Cj and C(j+2) in the **C** matrix, and have both integers j and j+2 prime, can result in Cj' containing a stalactite which reaches the l.d. This fact (which perhaps should have been established as a lemma) will allow us later to construct an algorithm for separating out (and counting) the potential twin primes which actually‘become’ twin primes, from the other potential twin primes which don’t. In the next subsection, we shall formalize this and other facts that we shall need to produce the algorithm.

11.3 The Sieving Process in the Matrix C′

We next proceed to carry out a sieving process, analogous to that used on matrix C', to determine which columns have stalactites which continue to ‘grow’ with 1s, down to the leading diagonal. Those that do reach the l.d. will indicate where the twin primes occur in the columns of **C'**.

The process is to compute a new matrix, whose rows are comprised of elements of the following sequence **1', 1'** ∧ **2', 1'**∧**2'**∧**3', ...** with general term the BP-factorial of **n'.** In the final matrix, which we designate by **PBPS(C'),** the columns with stalactites reaching the l.d. will indicate where the twin primes are.

12. COMPUTING THE PARTIAL BOOLEAN PRODUCT SEQUENCE OF C′

We first show how rows 1 to 3 of the **PBPS**(**C′**) matrix are computed, working with $\mathbf{C'_{12}}$ as example.

12.1 The first three rows of PBPS(C'_{12})

(1) **row 1'** of the **PBPS** matrix is the same as row **1**' of the starting matrix.

(2) we compute **row 2'** of the **PBPS** matrix thus:

row **1'**		1 1 1 1 1 1 1 1 1 1 1 1 ...
row **2'**	$\wedge$	1 0 1 0 1 0 1 0 1 0 1 0 ...
		-----------------------------------
joint row, **new r_2**		1 0 1 0 1 0 1 0 1 0 1 0
pTP cols. j'		1 3 5 7 9 11

(3) we compute **row 3'** of the **PBPS** matrix thus:

new row 2		1 0 1 0 1 0 1 0 1 0 1 0
old row 3	$\wedge$	0 1 0 0 1 0 0 1 0 0 1 0
		-----------------------------------
joint row, **new r_3**		0 0 0 0 1 0 0 0 0 0 1 0
pTP cols. *j*		5 0 11

*Figure 12.1 Computation of first three rows of **PBPS(C'_{12})***

Note how pTP columns are occurring ('picked out', as it were) in the new rows: In the new row r_2 all of colums 1,3,5,9,11 are pTPs. Whereas, in the new row r_3 only columns 5 and 11 are pTPs. Hence the **3'-**sieve has stopped the stalactites in columns **1'**, **3'.**

The $\begin{pmatrix}1\\0\end{pmatrix}$ in column 3 of (3) indicates that a stalactite has been stopped on the l.d. and so c_3 is TP.

Lemma: The **3'-sieve** viz. 010,010,010,010 ... , has 0-ordinals 1,3,4,6,7,9,10,12, etc. So the sieve stops all stalactites in columns with odd subscripts in this sequence , in row **R3'**. Those stalactites in even-subscripted columns were all stopped in row **R2'**. We note that the odd-subscripted columns separate into two sequences, viz.

(i) columns 7', 13', 19', etc having column subscripts in the A.P.(7;6).

and (ii) columns with subscripts in the A.P.(3;6)

Note: The sequence in the A.P. of (i) is all of upper-6 numbers, which we already know cannot indicate a twin prime from **C';** by construction, they all contain a factor of type 6m+3, and hence all are measurable by 3. (as the **3'**-sieve has shown us).

12.2 The re-cast of Figure 12.1 below shows again, with different details, how the fundamental cycles of 1,2,3 of **PBPS(C′)** are obtained.

(1) row 1 of the **PBPS(C′)** matrix is the cycle-number **<u>1</u>** , using its f.c. **$\underline{1}^{(1)}$,** in **C'**.

(2) we compute r_2 of the **PBPS(C′)** matrix from the f.c. of **<u>1</u>** $\wedge$ **<u>2</u>** (see thm. 6.3(ii))

2 cycles of the f.c. of **<u>1</u>**	1 1	
1 cycles of the f.c. of **<u>2</u>**	1 0	(Boolean Product)
	----------	
new r_2: joint f.c. **$(\underline{1} \wedge \underline{2})^{(1)}$**	1 0	(period 1x2 = 2)
stalactite cols. *j*	1	

(3) we compute r_3 of the **PBPS(C′)** matrix thus:

3 cycles of the new r_2	1 0 1 0 1 0	
2 cycles of the f.c. of <u>**3**</u>	0 1 0 0 1 0	(Boolean Product)
	------------------	
new r_3 joint f.c. **(<u>2</u> ∧ <u>3</u>)**[(1)]	0 0 0 0 1 0	(period 2x3 = 6)
stalactite cols. *j*	5	

Figure 12.2 Computation of first three fundamental cycles in rows of ***PBPS(C′)***

The following diagram shows matrix **PBPS(C'_{30})**, in skeleton form, with all the 1s shown, but only those 0s which stop stalactites are shown. All the blank elements are 0s.This figure should be compared with Figure 7.2, which showed **PBPS(C_{30})**. There we called the stalactites the prime ribs of the natural numbers. Now we can similarly call the stalactites in **PBPS(C'_{30})** the 'joint-twin prime ribs of the natural numbers'.

Note that each stalactite reaches the l.d. in its column, and so indicates the left-arm of a CP. In (1,30) the algorithm has 'found' the CP's (3,5), (5,7), (11,13), (29,31). The complete set of them has been found, each in turn, overstepping the non-TP's in the process.

Note for future reference that all pTP's, except in c'_5 and c'_{23}, were stopped by primes less than 5.
Note also the triangle shown between row 17 and row 29. These triangles continue thus, down the l.d. from TP to TP.

	0									1										2										3
S'ₙ\ t	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	P	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	0		T		1		0		0		1		0		0		1		0		0		1		0		0		1	
4					1						1						1						1						1	
5					T						1						1						0						1	
6											1						1												1	
7											1						1												1	
8											1						1												1	
9											1						1												1	
10											1						1												1	
11											T						1												1	
12																	1												1	
13																	1												1	
14																	1												1	
15																	1												1	
16																	1					→							1	
17																	T	0	0	0	0	0	0	0	0	0	0	0	1	
18																		0											1	
19																			0										1	
20																				0									1	
21																					0								1	

22		0						↓	1	
23			0						1	
24				0					1	
25					0				1	
26						0			1	
27							0		1	
28								0	1	
29									T	
30	0								0	0

*Figure 12.3 Che **PBPS (T'$_{30}$)** matrix*

13. PROPERTIES OF BLOCKS IN MATRIX C′

13.1 As in Section 8, where Blocks in matrix **C** were defined and studied, we can define primorial Blocks in the matrix **C′.** In the following sub-sections we show how these blocks are defined, and how we study them in the same way as in Section 8 for matrix **C.** We can then compare the results from **C** and **C**' , and discuss the relative meanings of the frequencies of *f* and *g* 2-vecs, with regard to occurrences of primes and twin primes.

13.2 On Boolean row-products of Blocks in C'

We begin this part by examining the changes in the formulae for row-products of the blocks B(p_k, X_k) in the matrix **C′**, as distinct from those in **C**. In particular, we need to know the frequencies of the (2 x 1) vectors from the last two columns of the blocks.

Theorem 13.1: (see Section 8) The general formulae for #d, #e, #f and #g from the last two rows, **S'$_{k-1}$** and **S'$_k$,** in the block B(p_k, **X_k**) in matrix **C′** are::

__General formulae__: #d $\equiv d_k = 2(X_{k-1} - X_{k-1}^{(-2)})$, where $X_k = \prod_{n=1}^{k} p_n$,

and $X_k^{(-2)} = \prod_{n=1}^{k}(p_n - 2)$, defined as 1 when k = 1

#e $\equiv e_k = (p_k - 2)(X_{k-1} - X_{k-1}^{(-2)})$;

#f $\equiv f_k = 2g_{k-1} = 2X_{k-1}^{(-2)}$;

#g $\equiv g_k = X_k^{(-2)}$.

Proof: This follows directly from the method of generating these frequencies, as follows. Using induction, assume the above formulae to be true from *k=2* to *k=k-1* (by observation they are true for k=2)

Using the methods of theorem 8.2, we have to find first the generating function for row r_{k-1} , which by the assumption, is $u_0 + u_1 i$, with $u_0 = X_{k-1} - X_{k-1}^{(-2)}$ and $u_1 = X_{k-1}^{(-2)}$.

Then we find the generating function for r_k , which is $v_0 + v_1 j$, $v_0 = 2$ and $v_1 = p_k - 2$.

Applying Theorem 8.2, we find that $d_k = u_0 \,.\, v_0 = 2(X_{k-1} - X_{k-1}^{(-2)})$,

$e_k = u_0 \,.\, v_1 = (p_k - 2)(X_{k-1} - X_{k-1}^{(-2)})$,

$$f_k = u_1 \cdot v_0 = 2X_{k-1}^{(-2)},$$
$$g_k = u_1 \cdot v_1 = X_{k-1}^{(-2)} .$$

These are each in accord with its formula given above. Hence by induction, the formulae are true for all values of k > 1. □

The table below shows the calculations for p_k with k = 1 to 5.
The generating function for the *k*th *t*-prime *f.c.* (the t-sieve) is 2 + (p_k – 2)*j*.
Let us call this G_k. In the third column of the table in Figure 11.4 we give G_k for k = 1, ..., 5. In the final four columns, we list the corresponding frequencies of *d,e,f,g* which occur in the fundamental cycles of **$\underline{p_k}$** . Recall that *d,e,f,g,* are respectively $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$ occurring in rows $\begin{pmatrix}r_{k-1}\\r_k\end{pmatrix}$ of **PBPS**(**C′**): hence they are calculated from the sequence of computations of

k	p_k	G(**S'**$_{k-1}$)	G_k	#d	#e	#f	#g	#g/#f
1	2	0 + 1*i*	1 + 1*j*	0	0	1	1	1
2	3	1 + 1*i*	2 + 1*j*	2	1	2	1	0.5
3	5	5 + 1*i*	2 + 3*j*	10	15	2	3	1.5
4	7	27 + 3*i*	2 + 5*j*	54	135	6	15	2.5
5	11	195 + 15*i*	2 + 9*j*	390	1755	30	135	4.5

Formula for the last column: $\dfrac{\#g}{\#f} = \dfrac{1}{2}(p_k - 2)$ for k > 1 .

Figure 13.1 Table for d,e,f,g frequencies in prime row f.c .s of ***PBPS(C′)***

13.2 **On the cycling properties of Blocks in C′**

Properties of Blocks in matrix **C** were studied in Section **8.** The reader may wish to review that Section now. Much of what was said there can equally be said about Blocks in **C′**. We shall repeat theorems 8.2 and 8.3 here, translating them so they apply in matrix **C′**.

Then we shall give a full treatment of the process of finding the last row of blocks B(p_k, X_k) for k = 2,3,4 using tabulations of the final two rows of PBPS(**C′**) to demonstrate.

13.3 The Blocks in C′

Theorem 13.2: (see Section 8 for Blocks in **C**)
Let B(*r,s*) be the rectangle (or block) consisting of the following set of elements in **C′** : $\{m_{i,j} \mid i = 1, ... ,r ; j = 1, ... ,s \}$
The following Blocks repeat consecutively and indefinitely to the right in **C′:**

(i) $B(2,2) = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$.

Proof: **$\underline{1}$** and **$\underline{2}$** cycle jointly with period 2.

$$\textbf{(ii)}\quad B(4,6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = B(p_3 - 1 \,,\, X_2)$$

Proof: **2** and **3** cycle jointly with period 2x3 = 6.
And **4** ≡ **2** doesn't affect their joint cycle-pattern.

Note that if the Boolean product of the rows of B(4,6) be taken, the resulting vector is (0 0 0 0 1 0) is equal to $\underline{\mathbf{6}}^{(1)}$ **,** the f.c. of the joint cycle of the two prime rows in the Block.

Theorem 13.3:
The Blocks $B(p_{k+1}\text{-}1, X_k)$, $k = 1,2,3, ...$ repeat consecutively and indefinitely to the right in **C′**.

Proof: The prime cycle-numbers in $p_1, p_2, ... , p_k$ cycle jointly with period X_k. Inclusion of any non-prime cycle-numbers, in other rows r_i with $i < p_{k+1}$, does not affect their joint cycle-pattern (the 'triangle squeeze' argument in Theorem 6.5 makes it clear why not). □

Note that the Boolean and of the rows in the Block is the f.c. of the joint cycle of all the prime cycle-numbers amongst the rows of the Block.

13.4 Tabulation of Block data for p_k with k=3,5,7,11

The following table shows how the last rows of $B(p_k , X_k)$ are calculated for k=3,5,7. In particular, the case k = 4, with Block B(7, 210) should be well-studied, to see how the sieving procedure continues to pass on, down to the last row, the next twin prime, first (11, 13), and then all the subsequent potential TPs in that final block row. And to see what information we glean from the final two rows, and their 2-vecs.

Figure 13.2 Book-keeping of TPs and pTPs occurring in last two rows of Blocks

B(3,6)		**1 2 3 4 5 6**	
row 2	$3^*\underline{\mathbf{2}}^{(1)}$	1 0 1 0 1 0	row 2 shows three pTPs (row 1 is 1 1 1 1 1 1)
row 3	$2^*\underline{\mathbf{3}}^{(1)}$	0 1 0 0 1 0	row 3 shows that (3,5) is a TP, and (5,7) is a pTP
In r_3	$(\underline{\mathbf{2}\wedge\mathbf{3}})^{(1)}$ ∧	0 0 T 0 1 0	This is the bottom row of the Block: It cycles to ∞

****

		0 1 2 3
B(5,30)		**1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
row 4	$5^*\underline{\mathbf{6}}^{(1)}$	0 0 0 0 1 0,0 0 0 0 1 0,0 0 0 0 1 0,0 0 0 0 1 0,0 0 0 0 1 0
row 5	$6^*\underline{\mathbf{5}}^{(1)}$	1 1 0 1 0,1 1 0 1 0,1 1 0 1 0,1 1 0 1 0,1 1 0 1 0,1 1 0 1 0
In r_5	$\underline{\mathbf{30}}^{(1)}$ ∧	0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0
		f e g g f g

The first f shows that c'_5 is T. The e shows c'_7 is nT. The g s show that c'_{11}, c'_{17}, c'_{29} are all pTs. The f on c'_{23} shows that its stalactite is stopped there, and is now nT (c'_{25} was stopped in row 3, since $c'_{25} = c_{25} \wedge c_{27}$, and c_{27} is stopped in row 3 of **C**.)

****

0 1 2 3

B(7,210) (1st 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(1)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
cycling *$7^{(1)}$ 1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
d g g g

Note the d shows that 7-stal was stopped earlier, The first g shows a pC'-prime, which must continue down to the l.d., and become the C' (11,13). Note that the 13-stal has also been stopped earlier, so the algorithm will next stop 11, and show 17 as the next pT.

---

3 4 5 6

B(7,210) (2nd 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(2)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0
cycling *$7^{(1)}$ 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1,
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0,
g f g

---

6 7 8 9

B(7,210) (3rd 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(3)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0
cycling *$7^{(1)}$ 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0 1 1 1 1 0 1,
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0,
g f f

---

9 10 11 12

B(7,210) (4th 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(4)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
cycling *$7^{(1)}$ 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0,
g g f

---

12 13 14 15

B(7,210) (5th 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(5)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
cycling *$7^{(1)}$ 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
f g g

---

15 16 17 18

B(7,210) (6th 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**
$30^{(6)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
cycling *$7^{(1)}$ 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0
In r_7 $210^{(1)}$ ∧ 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,
f g g

18 19 20 21

B(7,210) (7th 30) **1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0**

$30^{(7)}$ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,

cycling **$7^{(1)}$** 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,1 1 1 1 0 1 0,

In r_7 **$210^{(1)}$** ∧ 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0,

g g g

****

The above figure charts the whole sequence of the penultimate row, the 7-sieve in action, and the final row, of B(7,210) displayed in seven sections each of length 30.

The following notes explain how the stalactite algorithm has worked to this level:
(i) In $30^{(1)}$ the *d* in col. 7 shows that **7'** is non-TP. The three *g* s indicate three pT s, correctly of course (though we do not need to know their corresponding natural numbers at this point). All the three pT's have grown their stalactites down, through the prime sieves 2,3,5,7, and inevitably also through the non-prime sieves 4 and 6 too.
(ii) All of the following *g* s indicate pTPs of length 7. There are 15 of them.
(iii) All of the following *f* s indicate a stopped stalactite, and hence a non-TP column. They occur in columns numbered: 47, 77, 89, 119, 131, 161. Each can be explained in terms of multiples of 7 which involve, in sequence, the primes 7 and higher. The lower prime sieves have already done their work on the stalactites, in rows before r_7 . Thus: (47,49) fails because 49 = 7 x 7; (77, 79) has 77 = 7 x 11 ; (89,91) has 91 = 7 x 13; (119,121) has 119 = 7 x 17 and 121= 11x11; (131,133) has 133 = 7 x 19; and finally (161,163) has 7 x 23. Summarizing, the 7-sieve allows 15 stalactites to pass through (in the *g* cols.) and remain pT's, and stops 6 stalactites (in the *f* cols.), being in cols. 7 x (7, 11, 13, 17, 19, 23). The only other possible candidate for stoppage in the interval (7,210) is 7 x 23 = 203, but (203,205) has right-arm = 5 x 41, so was stopped by the 5-sieve.
(iv) All but two of the remaining pTP's will proceed down to the l.d. and register as TP's. The ones that are stopped before then are (167,169), stopped by the 13-sieve; and (209, 211) since 209 = 11 x 19 will not pass the 11-sieve.
(v) We emphasize, for the 3rd or 4th time, that the important thing about the progression of this algorithm through the B-blocks is not about the sequence of twin primes per se, but the sequence of *f* s and *g* s which indicate them. And because we have shown by formulae in Theorem 10.1 , exemplified down to row 7 in Figure 10.1, that the frequencies #*f* and #*g* increase indefinitely within the cycling blocks $B(p_k, X_k)$, with #*f* always less than #*g*, we have demonstrated that the production of twin primes can never cease. (Turner believes this is proof of the TPC. However, we say more about the whole process of the proof in the next section, adding more evidence, in a more formal manner.)

Further tabulation of the sieving process, for intervals (7, 210) and (11, 2310)

Clearly, as p gets larger, X_p gets rapidly larger, and it would seem we could not usefully display further tables of *f* and *g* frequencies and discuss their consequences. However, we shall present two more such tables, this time extracting just the *f* and *g* columns for the blocks $B(7, X_7)$ and $B(11, X_{11})$.

Turner applied the sieving process to the second Block, using only pencil and paper, and no electronic calculator, discovering from the total of 2310 columns all of the 30 *f*-columns and 135 *g*-columns in less than 6 hours. Some of this time was spent, of course, on delighting in the many symmetries and confirming patterns which he discovered along the way. Above all, he found that he could rely completely upon the powerful Palindrome Principle that governs the sequences of both the f-columns and the g-columns. It is only necessary to discover their columns for the first half of the journey (up to C1155) and then proceed to compute the palindromic complements in 2308. For examples, the first three *g*–columns (pTPs) are17, 29, 41 (actual TPs). and their complements from the other end of the Block of columns, are respectively 2291, 2279, and 2267 (obtained by subtracting from 2308).

Figure 13.3 The f and g sequence through the eleven 210-cycles in B(11,2310)

	#f	#g	In 1st seventy	In 2nd seventy	In 3rd seventy
$210^{(1)}$	2	13	f g g g g	g g g g	g g g g g f
$210^{(2)}$	3	12	g g g f g	g g f g	g g g g f g
$210^{(3)}$	1	14	g g f g g	g g g g	g g g g g g
$210^{(4)}$	4	11	g f g f g	g g f g	f g g g g g
$210^{(5)}$	3	12	g g f g g	f g g f	g g g g g g
$210^{(6)}$	4	11	g f g g f	g g g g	f g g f g g
$210^{(7)}$	3	12	g g g g g	f g g f	g g f g g g
$210^{(8)}$	4	11	g g g g f	g f g g	g f g f g g
$210^{(9)}$	1	14	g g g g g	g g g g	g g f g g g
$210^{(10)}$	4	11	f g g g g	g f g g	g f g g g f
$210^{(11)}$	1	14	g g g g g	g g g g	g g g g f (g)

Comments to be inserted later, together with the table for B(7,210)

Note immediately, however, that the table show how the *f*s and *g*s satisfy the palandrome principle (leave out the final *g*), and how evenly spread the *g*s (which indicate pPs) are spread across the batches of 30 columns.

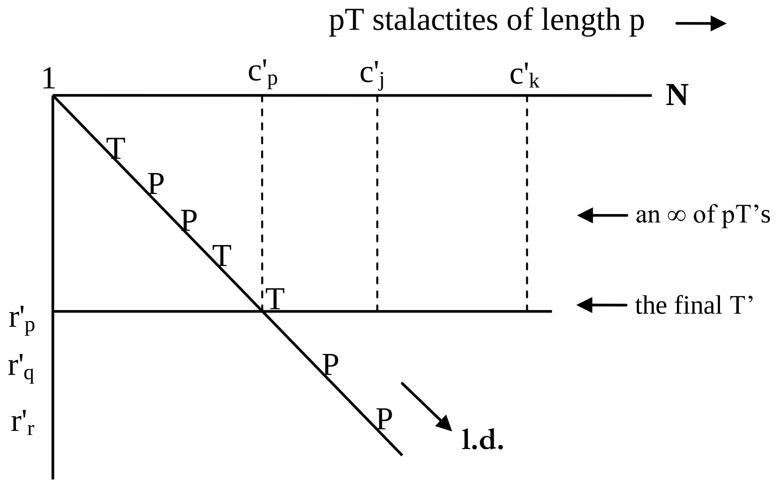
14. THE SEQUENCE OF TWIN PRIMES IS INFINITE

We have now developed the ideas, examples and theorems necessary for us to prove the twin primes conjecture. Indeed, we have already given all the evidence which clinches the proof. However, we shall go through all our arguments yet again, putting them in their proper sequence, and using a diagram of **PBPS(C′)** to help elucidate matters.

Theorem 14.1 (the TPC):
There is an infinity of twin primes in the sequence of natural numbers.

Proof: (see Fig. 14.1 below)

Figure 14.1 Skeleton diagram of ***PBPS(C')***



Proof of the TPC:

The diagram illustrates how the **PBPS**(**C′**)-sequence algorithm produces a sequence of Ts and Ps down the leading diagonal l.d. To prove the TPC conjecture, we assume first that it is false. In which case there must be a final twin prime T. In the diagram we assume this occurs in row r'_p and column c'_p.

Then we know by Block-cycling (see Thm. 11.3) that there is an infinity of pTP's after column c'_p. We have shown the first two of these in columns c'_j and c'_k .

Now the final 0 in the j-sieve occurs in columns j, 2j, 3j, etc., so until it reaches column j^2 it cannot stop any pT, since all stalactites in those columns will have already been stopped by an earlier prime sieve. Unless, of course, it stopped a stalactite in c'_j, in which case a new twin prime would be created, and the assumption we began with would be discredited. [to understand all these statements, see Figure 11.5 for examples and read the notes at the end of B(5,X_5), and (i), (ii), (iii) after B(7,X_7)]. Beyond column j^2 the j-sieve presents 0s to columns j(j+1), j(j+2), j(j+3), etc., and again (until j^3 is reached) the stalactites in these columns will have been stopped by a prime sieve earlier than the j-sieve.

Since the j-sieve is cycling out of phase with the last row of B(p,X_p), we know that it cannot stop all of the stalactites which have passed through the (j-1)-sieve (i.e. the p-sieve in row r_p.), not even within the first cycle of the block, say $B_p^{(1)}$.

Synopsis of the Proof:

Our proof has been completed in several stages**,** summarized as follows. Supporting references and discussions on each have been given above, together with proofs when necessary; more discussion and elucidation is given below, including a treatment of gnomons in C'_p matrices**:**

(i) A matrix **C′** (the twin primes matrix) is derived from **C** (the primes matrix) using the formula $t_{i,j} = m_{i,j} \wedge m_{i,j+2} \; \forall i, j \; \varepsilon \; N$.

We can view it as being formed sequentially from $\mathbf{C'_1 = C_1}$, with a gnomon being added to each in turn, as we did with the **C**-sequence. We discuss the gnomons in (ii) to (vii) below.

(ii) Let p be a prime integer, so it corresponds to **p̲** in **C** and to **p̲'** in **C′**

The p-gnomon in $\mathbf{C_p}$ has:

(p-1) 1s in its lower arm,

(p-1) 1s in its right arm (hence c'_p has an unstopped stalactite),

the corner element is 0.

The stalactite in c'_p is stopped at the l.d., whence the p-gnomon is prime.

(iii) In matrix $\mathbf{C'}$, however the gnomons corresponding to the primes in $\mathbf{C}$ are different. Indeed there are two types of prime p-gnomon in $\mathbf{C'}$, which we describe below. First we point out that $\mathbf{C'}$ is not symmetric, so its rows differ from its columns.

Each row r'_n cycles, with an f.c. of length n, same as in $\mathbf{C}$.

The columns of $\mathbf{C'}$ cycle too, but their periods vary quite wildly. The period of column c'_j is equal to the length of $(c_j \wedge c_{j+2})^{(1)}$.

(iv) Let a p-gnomon in $\mathbf{C'}$ be the gnomon of matrix $\mathbf{C'_p}$ with p a prime.

gnomon: Then its lower arm plus corner element has

pattern 1 1 1 ,,, 1 0 1 0, having (p-2) 1s and 2 0s. But its right arm is different (see (v))

[N.B. We have adopted the convention of using a prime-symbol to mark rows and columns of **C′.]**

(v) Since $c'_p = c_p \wedge c_{p+2}$; it is immediately plain that c'_p will contain a column of *(p-1)* 1s if and only if the col. pair (c_p, c_{p+2}) is a twin prime pair in $\mathbf{C}$. In that event, we shall say that $\mathbf{C'_p}$ has a TP-gnomon.

(vi) If (c_p, c_{p+2}) is not a twin prime pair, then the first p-1 elements of c'_p will be the same as those of c_{p+2} , and there will be 0s amongst them, since $p+2$ is non-prime. The stalactite in C'_p will have been stopped before reaching the l.d., and then we have a non-TP, or nTP-gnomon.

(vii) Thus there are two types of p-gnomon in $\mathbf{C'_p}$ matrices, TP ones and nTP ones.

(viii) It is shown that the unstopped stalactites to the right of the l.d., that can occur as the partial Boolean row 'and' matrix **(PBPS**(**C′**)) is computed sequentially from the rows of **C′**, are in a sequence of columns (say $\mathbf{C} = \{c'_k\}$) such that for all k in that sequence, the pair (k, k+2) is either a twin prime (TP) or a potential twin prime (a pTP). We may write that c'_k is either TP or pTP; or, more simply, either T or pT.

(ix) The next stage of the proof of TPC is to show that the sequential process of computing the rows of **PBPS**(**C′**), applying the sieve of the f.c. of $\underline{\mathbf{p}}$, viz. $\underline{\mathbf{p}}^{(1)}$ in r_p, to the cycling pattern in row r'_{p-1} , will always begin with a sequence of (0,1)-vectors in

rows $\begin{pmatrix} r_{p-1} \\ r_p \end{pmatrix}$ of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \cdots$ up to c'_{p-1} , and then:

either (a) complete a TP-gnomon R-arm in c'_p , where a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ vector occurs,

or (b) complete an nTP-gnomon R-arm in c'_p, where a $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ vector occurs.

In the case (a) a new twin prime has been 'found' or 'created'.

(x) In both cases of (ix), proceeding along the rows to the right, a sequence of vectors $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ occurs, until a vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ occurs, in c'_k say. This indicates that c'_k is pTP.

(xi) In both cases of (ix) and (x), we shall show that the pTP stalactite in c'_k must keep on growing until it reaches the l.d., and hence proclaim another TP.

(xii) Finally, using Theorem 10.1, we supply a proof that this algorithm must continue indefinitely, no matter how far we travel towards infinity along the number line. That means, by (ix), that there is an infinity of twin primes.

The above stages and supporting material complete our proof of the TPC.

Crucial Points:

Once again, we urge the reader to note that at no point in the above synopsis do we refer to actual numbers, primes or twin primes. In our earlier discussions we have given plenty of examples of these, in relation to the algorithm developments, in order to demonstrate that the algorithms do actually work. But we stress that we claim no ability to predict the whereabouts of primes and twin primes. We can determine only how they will occur in relation to the elements of Arithmetic Progressions of potential Ps and T's,

Throughout, we work with occurrences of (0,1)-patterns, locations of which are capable of being predicted precisely in terms of A.P.s, because of the cycling of the patterns in blocks $B(p_i,X_i)$, the cycling of the cycle-numbers themselves, and the notions of 'coprimeness' (to coin a term) , and potential primes (pPs) and potential twin primes (pTPs), in doubly-cycling matrices **C** and **C′**.

The patterns of the primes can be discerned through the patterns of coprimeness relations. Or (to coin an aphorism) with our approach, *coprimeness begets primeness.* We can define a measure of the *coprimeness of a number n* to be $\phi(n)/n$ *;* and then define *n* to be prime if its coprimeness is maximal. This is a different point of view from the minimalist one whereby *n* is prime if it has fewest possible factors. Both views see the same object, of course (compare a positive photograph of a scene, and its complementary negative – can one be declared to be a superior view of the scene than the other?)

15. ON THE MUSIC OF INTEGERS AND PRIMES

In this final section of the paper, we wish to comment on analogies that we can draw between musical melodies and chords, and polyphonic music, and the notion that similar 'musical sounds' can be 'heard' in the 'vibrations and relative vibrations' of the integers themselves. Much has been written on this kind of analogy, in the long history (and worship) of the natural numbers and their properties, which writing extends through three millennia and more.

In his book entitled 'Music of the Primes' [7], David Wells makes much of the music he hears from the study of primes in relation to the Zeta-function. We believe that there is much more call for listening to the music made by the integer cycle-numbers, as their cycling and joint-cycling in the matrix **C** proceeds, not only along the rows, but also down the columns: and in many other ways too. The very action of sieving by a prime-sieve can be viewed as a kind of compression wave, here and there stopping stalactites, and creating primes on the leading diagonal (with resounding crashes of cymbals?). History books tell us that Pythagoras, who invented the science of music, and then claimed to hear the music of the spheres which maintained credence for two millennia as a part of his theory of the cosmos, was the only mortal who actually could hear it (but, after all, he was also a half -son of the god Apollo; that would help!). If the glass sphere music and the Zeta function music can be heard, Turner has no qualms in claiming to hear a veritable symphony of numbers emanating from his cycle-number matrix.

Glossary of terms and characters (symbols)

Terms	Symbol, Acronym	Page Number
cycle-number	$\underline{n}$	
fundamental cycle	$\underline{n}^{(1)}$, f.c.	
enteger	e	
enteger word, (e-word)	$\underline{e}$	
kappa operator (on e or **$\underline{e}$**)	κ	
cycle-number matrix	**C**	
element of **C**	$m_{i,j}$	
*i*th row of **C**	r_i	
*j*th column of **C**	c_j	
twin prime matrix	**C'**	
element of **C'**	$c'_{i,j}$	
*i*th row of **C'**	r'_i	
*j*th column of **C'**	c'_j	
stalactite in column *j*	*j*-stal , c_j-stal , c'_j-stal.	
leading diagonal of a matrix	l.d.	
potential prime	pP	
potential twin prime	pTP or pT	
unstopped stalactite in **C**	pP	
stopped stalactite in **C**	P (if to l.d.) , nP (otherwise)	
Partial Boolean Row-Sequence matrix	PBPS(**C**) , PBPS(**C'**)	
Block matrix (sub-matrix of **C** or **C'**)	B(p_i , X_i)	
p_1x p_2 x p_3 x ... x p_i *(primorial p_i)*	X_i (or p_i#) (c.f. factorial n)	
(p_1-1)(p_2-1)...(p_i-1)	X_i^-	
(p_1-2)(p_2-2)...(p_i-2)	X_i^{-2}	
Product of all distinct prime factors of n	r(n) *(the radical of n)*	

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